Transaction costs, Shortselling Constraints and Taxes: A Unified Approach *

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First Version : December 1996
This version : November 1997

Abstract

In this paper, we propose a general model for discrete and finite set-up. First, we prove an absence of arbitrage result. Then, using this key result, we derive all the well-known no arbitrage theorems on imperfect markets, extending them to the case of assets paying dividends. We also prove some new results on taxes.

Keywords : Arbitrage, market’s imperfections, stochastic process, martingale, Farkas’s Lemma.

1 Introduction

The valuation of contingent claims is prominent in the theory of modern finance. It has been initiated by the well known works of Black-Scholes (1973) and Merton (1971). Later the papers of Harisson-Kreps (1979), Harisson-Pliska (1981), Kreps (1981) and Dalang-Morton-Willinger (1990) formalize the theory of pricing by arbitrage in frictionless markets. Loosely speaking, an arbitrage opportunity is a way to produce nonnegative wealth with

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positive expected value out of nothing. The fundamental Theorem of asset pricing (FTAP) state as follows: there is no arbitrage opportunity if and only if there exists an equivalent probability measure which turns the price process (appropriately renormalized) into a martingale. This result holds for a frictionless framework. An important body of literature has been developed to take frictions into account.

In the transaction costs framework, the fundamental idea was introduce by Bensaid-Lesne-Pagès-Scheinkman (1992): pricing by replication is not optimal. Instead of replicating strategies, they consider dominating ones and prove that they can be costless. Using this domination price concept, Jouini-Kallal (1995 a) provide an extension of the FTAP to the transaction costs case using a stronger version of the no-arbitrage concept: the no free lunch one. More precisely, they prove that the absence of free lunch is equivalent to the existence of at least an equivalent probability measure that transforms some processes between the bid and the ask price processes of traded securities into a martingale. In the shortselling constraints framework, Jouini-Kallal (1995 b) provided an extension of the FTAP. If short-selling is possible but costly, they show that a securities price system is arbitrage free if and only if there exist a numeraire and an equivalent probability measure for which the normalized (by the numeraire) price processes of traded securities that cannot be sold short are supermartingale and the normalized price processes of the securities that cannot be hold in nonpositive quantities are submartingale. Schürger (1996) proves an extension of the FTAP considering the no-arbitrage condition and he obtains the existence of \( \tau \)-martingale measure. Finally, in the taxe’s framework, Ross (1987) extends the Arbitrage Pricing Theory (1976) to taxe’s set-up. He considers a local version of arbitrage, and under technical conditions he obtains an absence of arbitrage theorem. Nevertheless, he studies only a two period model, and the problem becomes much more complicate in a multiperiod framework. The tax payment value depends on the whole history of investment.

In this paper, we focus on the martingale approach of the valuation by arbitrage and we study the previous different kind of frictions: transaction costs, no shortselling, shortselling costs and taxes. In section 2, we present a general model for discrete and finite financial markets. The key idea is to represent every asset by its cash-flow as in Dermody-Rockaffelar (1991, 1995), Cantor-Lippman (1985, 1995), Milne (1996), Adler-Gale (1997) and Carassus-Jouini (1995, 1997). We consider a large family of cash-flows, including the strategy’s randomness in the asset. For example, consider a stock described by the price process \( (S_t)_{0 \leq t \leq T} \), where \( T \) is the time horizon and paying dividends \( (D_t)_{1 \leq t \leq T-1} \). If an investor buys at time zero a share
of stock and and sells it at time $T$, the associated cash-flow sequence $\Phi$ will be: $\Phi_0 = -S_0$, $\Phi_t = D_t$ for all $t \in \{1, ..., T - 1\}$, and $\Phi_T = S_T$. Notice that we include the asset price in the cash flow sequence. Now considering all the possible buying and selling dates, we obtain a family of cash-flows, which represents the stock. In section 2, we prove an absence of arbitrage result for cash flows sequences, using an adapted version of Farkas’s Lemma. Then, in section 3, we apply this result to financial models. Successively, we study the case of complete and incomplete markets, shortselling constraints and transaction costs. We find again the literature’s classical results, extending them to the case of assets paying dividends. In addition of the unified approach, the proofs are particularly simple. Moreover, this approach allows us to treat other kind of imperfections, as taxes.

2 The general model

We work in a discrete and finite setting. We suppose that there is only a finite number of observations, indexed by the set $T = \{0, 1, ..., T\}$ of a financial world. This system can only be in a finite number of states, denoted by the set $\Omega$. The information is modeled by a filtration $\mathcal{F} = \{\mathcal{F}_t, t \in T\}$. We suppose that $\mathcal{F}_0 = \emptyset$ and that $\mathcal{F}_T$ contains the whole information.

Every asset is represented through its cash-flow. We suppose that the model contains $N$ assets. Each asset $\iota$ has a finite time horizon $T_\iota$, where $T_\iota$ is less or equal to $T$. The asset $\iota$ will be described by a stochastic process, $\Phi^\iota_t(\omega)$. More precisely, $\Phi^\iota_t(\omega)$ is the cash-flow received (positively or negatively) from the investment $\iota$ at time $t$ and in state $\omega$. We suppose that each $\Phi^\iota$ is adapted to the filtration $\mathcal{F}$.

The investor is allowed to subscribe in every nonnegative fraction of a cash-flow (shortselling constraints). A strategy is a sequence $(\lambda_\iota)_{\iota=0,...,N}$ of nonnegative numbers. This sequence is chosen at time zero, and does not contain any uncertainty. First, the investor chooses if she/he want or not to subscribe to a cash flow sequence. Then, she/he chooses the number of subscription to the selected sequence. Therefore, in this model the whole uncertainty is taken into account by considering a large family of cash flows, sufficiently large to cover all possible strategies choice. The payoff at time $t$, in state $\omega$ associated to the strategy $(\lambda_\iota)_{\iota=0,...,N}$ is equal to $\sum_{\iota=0}^{N} \lambda_\iota \Phi^\iota_t(\omega)$.

We will say that a cash flow $\Phi$ is nonnegative (resp. positive, equal to zero) if for all $(t, \omega) \in T \times \Omega$, $\Phi(t, \omega) \geq 0$ (resp. $\Phi(t, \omega) > 0$, $\Phi(t, \omega) = 0$). An arbitrage opportunity will be a possibility to choose a strategy leading to a nonnegative payoff at each date and in each state of the world, and to
a positive one in at least one date and one state of the world.

Let $E$ be the expectation operator under the uniform probability $\pi$, the next theorem characterizes the absence of arbitrage opportunity.

**Theorem 2.1** If the assets $\Phi^\iota$ cannot be sold short, the absence of arbitrage is equivalent to the existence of a positive process $H$ with $H_0 = 1$, such that for all asset $\iota$,

$$\sum_{t \in T} E[H_t \Phi^\iota_t] \leq 0.$$ 

Notice that Theorem 2.1 works without assuming the existence of a numéraire. In order to prove Theorem 2.1, we need an adapted version of Farkas’s Lemma, which proof is carried out later. Let us define

$$\mathbb{R}^\alpha_+ := \{x = (x_i) \in \mathbb{R}^\alpha / x_i \geq 0, \text{ for } i = 1, \ldots, \alpha\}$$

and

$$\mathbb{R}^\alpha_{++} := \{x = (x_i) \in \mathbb{R}^\alpha / x_i > 0, \text{ for } i = 1, \ldots, \alpha\}.$$ 

**Theorem 2.2** If $\mathcal{Z}$ is a set of vectors in $\mathbb{R}^\alpha$ then exactly one of the following two alternatives must occur.

1. There is a nonnegative linear combination of vectors of $\mathcal{Z}$ which belongs to $\mathbb{R}^\alpha_+$ and is not equal to zero.
2. There exists a vector of $\mathbb{R}^\alpha_{++}$ which makes a nonpositive scalar product with all elements of $\mathcal{Z}$.

**Proof of Theorem 2.1**:

Let $\mathcal{Z}$ be the set of $\mathcal{F}$-adapted processes $\Phi^\iota$. Thus, the set $\mathcal{Z}$ is included in $\mathbb{R}^{T \times \Omega}$, and $\mathcal{Z}$ can be identified to a sub-vectorspace of $\mathbb{R}^{(T+1)|\Omega|}$. The absence of arbitrage opportunities implies that the first alternative of Theorem 2.2 cannot hold. Hence, there exists a positive process $H^*$, such that for $\iota = 1, \ldots, N$,

$$\sum_{t \in T, \omega \in \Omega} H^*_t(\omega) \Phi^\iota_t(\omega) \leq 0.$$ 

Let $H_t = E[H^*_t / \mathcal{F}_t]$, $H$ is a positive $\mathcal{F}$-adapted process. Recalling that $\Phi^\iota$ is $\mathcal{F}$-adapted, we find that,

$$\sum_{t \in T} E[H_t \Phi^\iota_t] = \sum_{t \in T} E[H^*_t \Phi^\iota_t] = \frac{1}{|\Omega|} \sum_{t \in T, \omega \in \Omega} H^*_t(\omega) \Phi^\iota_t(\omega) \leq 0.$$ 

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The process $H$ is positive and we can normalize $H$ by the condition $H_0 = 1$. Conversely, suppose that there exists a positive process $H$ such that for $t = 1, \ldots, N$, $\sum_{t \in T} E[H_t \Phi^i]$ is nonpositive. Now, suppose that there exists an arbitrage opportunity. Let $(\lambda_i)$ be one of the nonnegative strategy leading to a nonnegative and non-zero payoff. If we consider the expected value of the product of this payoff with $H$, we find a positive number. It is equal to $\sum_{i=1}^N \lambda_i \sum_{t \in T} E[H_t \Phi^i]$, but this quantity is nonpositive by assumption.

In Theorem 2.1, we require that no investment can be sold. The world "sold" is unappropriated in this context because an investor facing a cash flow sequence have only two choices: subscribing or not. If she/he subscribes to the sequence, she/he must keep it until the end of the sequence. In this context, it is equivalent to sell an asset or to subscribe at the opposite of the asset's cash flow sequence. So if the shortselling assumption for some cash-flow sequence does no correspond to an economic reality, we can avoid it assuming that the model contains the opposite of the cash flow sequence. This is done in the following corollary.

**Corollary 2.1** If there is no shortselling constrains on the assets $\Phi^i$, the absence of arbitrage is equivalent to the existence of a positive process $H$ with $H_0 = 1$, such that for all asset $i$,

$$\sum_{t \in T} E[H_t \Phi^i] = 0.$$  

**Proof of Corollary 2.1:**
Apply Theorem 2.1 to the cash flow $\Phi^i$ and $-\Phi^i$.

**Proof of Theorem 2.2:**
We introduce the following notations: $Z = \{z_1, ..., z_p\}$, and $\text{vect}^+ Z = \{\sum_{i=1}^p \lambda_i z_i / \lambda_i \in R_+, i \in \{1, ..., p\}\}$. Suppose that the property 1 is not satisfied, then $\text{vect}^+ Z \cap R_+^n = \{0\}$.

Denoting by $\Delta^\alpha$ the simplex of $R^\alpha$, that is, $\Delta^\alpha = \{x = (x_i) \in R_+^n / \sum_{i=1}^n x_i = 1\}$, we find that $\text{vect}^+ Z \cap \Delta^\alpha = \emptyset$.

It is well known that the set $\text{vect}^+ Z$ is a closed, convex cone of $R^\alpha$ (see Farkas’s Lemma) and that the simplex is a convex, compact set of $R^\alpha$.  

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Hence using Minkowski Theorem, there exist \( h \in \mathbb{R}^a \) and \( b_1, b_2 \in \mathbb{R} \) such that,

\[
< h, z > \leq b_1 < b_2 \leq < h, s >, \quad \forall z \in \text{vect}^+ Z, \quad \forall s \in \Delta^a.
\]

Notice that if \( z \) belongs to \( \text{vect}^+ Z \), then for all positive \( \lambda \), \( \lambda z \) also belongs to \( \text{vect}^+ Z \), and therefore \( \lambda < h, z > \leq b_1 \). Thus, \( b_1 \) can be chosen equal to 0 and we obtain for all \( z \in \text{vect}^+ Z \) that \( < h, z > \leq 0 \). Now choosing successively for \( s \) the vectors of the canonical basis of \( \mathbb{R}^a \), we find that \( h \in \mathbb{R}^a_++ \), and the property 2 is proven.

It remains to prove that properties 1 and 2 are incompatible. To do that suppose that they occur simultaneously. For all \( i \) from 1 to \( p \), \( < h, z_i > \leq 0 \). Taking a linear combination with nonnegative coefficient, we find that \( < h, \sum_{i=1}^p \lambda_i z_i > \leq 0 \). Recalling that \( \sum_{i=1}^p \lambda_i z_i \in \mathbb{R}^a_++ \) and \( h \in \mathbb{R}^a_++ \), we find that \( < h, \sum_{i=1}^p \lambda_i z_i > > 0 \). This contradiction ends the proof of theorem 2.2.

3 Applications to imperfect markets

As previously done, we denote by \( T = \{0, 1, ..., T\} \) the set of dates and by \( \Omega \) the states of the world’s set. The information structure is given by the filtration \( \mathcal{F} = \{ \mathcal{F}_t, t \in T \} \). We assume that \( \mathcal{F}_0 \) is trivial and \( \mathcal{F}_T \) contains the whole information. The market contains \( N \) risky assets. We denote by \( S^i \) the real-valued and \( \mathcal{F} \)-adapted stochastic price process of asset \( i \). The random variable \( S^i_t(\omega) \) represents the market value of asset \( i \) at time \( t \), in the state of the world \( \omega \). The market also contains an \( (N+1)^{th} \) asset indexed by 0 and denoted by \( R \). This asset have positive returns between its buying date and its expiration date. Without loss of generality, we can choose \( R_0 = 1 \). We assume that there is no short selling constrains on asset \( R \).

**Remark 3.1** In particular, if there exists a lending and a borrowing rate which are equal, we can choose for \( R \) this common rate. More precisely, let \( r_{t+1} \) be the interest rate between time \( t \) and time \( t+1 \). We suppose that \( r_{t+1} \) is \( \mathcal{F}_t \)-measurable and \( r \) is a positive process. A particular case of asset \( R \) is given by \( \frac{R_{t+1}}{R_t} = 1 + r_{t+1} \). The absence of short selling constrains on \( R \) implies that one can borrow and lend at the same rate \( r \).

The key idea of the paper is to represent an asset by the cash-flows that it may generate. We will use simple strategies, that is strategies where there is only one buying date and one selling date. If an investor want to follow a strategy involving several dates of trading, she/he will make linear
combinations with nonnegative coefficients of simple strategies. We will give later some examples. The set $I$, which inedexs the set of strategies, will contain the following informations. It points out the considered asset $i$, a date $t$ and an event $A \in F_t$. The event $A$ conditions buying and selling times $\tau_1$ and $\tau_2$, which are stopping times occurring after $t$. More precisely, let

$$I = \{i = (i, t, A, \tau_1, \tau_2) / i = 1, \ldots, N, t \in T, A \in F_t, (\tau_1, \tau_2) \in S_{t,T} \times S_{t,T}\},$$

where $S_{t,T}$ is the class of stopping time $\tau$, such that $t \leq \tau \leq T$. A simple strategy $(\lambda_i)_{i=(i, t, A, \tau_1, \tau_2) \in I}$ is a sequence of non negative number $\lambda_i$. Each $\lambda_i$ represents the number of asset $i$ bought at time $\tau_1$ if the event $A \in F_t$ occurs and sold at $\tau_2$ always if $A$ occurs. We will see that if there is no shortselling constraints, it is not necessary to precise if the buying date takes place before the selling date or not. Taking into account the new investments $i$ instead of $i$ allows us to transfer the randomness of the strategy in the classical sense in the investment.

In the next section, we will precise the exact form of the investments $(\Phi^i)_{i \in I}$ depending from the imperfection under consideration.

As in the previous section, the payoff associated to a strategy $(\lambda^i)_{i \in I}$ is equal to $\sum_{i \in I} \lambda_i \Phi^i$. We keep the same definition for an arbitrage opportunity, i.e. the choice of a strategy leading to a nonnegative payoff, which is positive in at least one date and one state of the world.

First, we present a consequence of Theorem 2.1 to the particular case of a financial market containing the asset $R$. The representation of asset $R$ is the following : let $i = (0, t, A, \tau_1, \tau_2) \in I$,

$$\Phi^i_t(\omega) = -R_t(\omega)I_A(\omega)I_{\tau_1}(t, \omega) + R_t(\omega)I_A(\omega)I_{\tau_2}(t, \omega).$$

We have used the classical notations : $I_A(\omega) = 1$ if $\omega \in A$, and zero else; $I_{\tau_1}(t, \omega) = 1$ if $\tau_1(\omega) = t$, and zero elsewhere. The cash-flow $\Phi^i$ corresponds to the following strategy : buy a unit of asset $i$ at $\tau_1$ if the event $A$ occurs and sell it at $\tau_2$ always if $A$ occurs. The absence of shortselling constraints on $R$ implies that $-\Phi^{(0,t,A,\tau_1,\tau_2)}$ also belongs to the model. Note that $-\Phi^{(0,t,A,\tau_1,\tau_2)} = \Phi^{(0,t,A,\tau_2,\tau_1)}$ and the absence of shortselling constrains appears then to be equivalent to the absence of ordering conditions between the buying and the selling date. Recall that the particular assumption on the returns of $R$ implies that $R$ is positive.

**Theorem 3.1** If the market contains an asset $R$ with positive returns and if the assets $\Phi^i$ can not be sold short, the absence of arbitrage is equivalent
to the existence of an equivalent probability \( \pi' \), such that, that for all asset \( i \), \( \sum_{t \in T} E_{\pi'}[\Phi^i_t R_t] \) is nonpositive.

We deduce immediately the following corollary.

**Corollary 3.1** If there are no shortselling constraints, the absence of arbitrage is equivalent to the existence of an equivalent probability \( \pi' \), such that, that for all asset \( i \), \( \sum_{t \in T} E_{\pi'}[\Phi^i_t R_t] \) is equal to zero.

**Proof of Theorem 3.1**

Applying Theorem 2.1 to \( \Phi^{0,0,0,T} \) (buy one unit of \( R \) at time 0 and sell it at time \( T \)) and to \( \Phi^{0,0,0,T,0} \) (sell one unit of asset \( R \) at time 0 and buy it at time \( T \)), we find that \( 1 = E[H_0] = E[H_T R_T] \). Let \( \pi' \) be the probability defined by \( \pi'(A) = E[H_T R_T I_A] \), for all \( A \in \mathcal{F}_T \). The process \( H \) and \( R \) are positive and thus \( \pi' \) and \( \pi \) are equivalent.

Now applying Theorem 2.1 to \( \Phi^{0,t,A,t,T} \) (buy of one unit of \( R \) at \( t \) if \( A \in \mathcal{F}_t \) and sell it at \( T \) if \( A \)), and to its opposite, we find that \( E[(R_t H_t - R_T H_T I_A)] = 0 \). From the arbitrariness of \( A \) in \( \mathcal{F}_t \), we obtain that \( E[R_T H_T / \mathcal{F}_t] = R_t H_t \). Now consider an investment \( \iota \in \mathcal{I} \),

\[
\sum_{t \in T} E[\Phi^\iota_t H_t] = \sum_{t \in T} E[\Phi^\iota_t E[R_T H_T / \mathcal{F}_t]]
\]

\[
= \sum_{t \in T} E[\Phi^\iota_t R_T H_T], \text{ because } \Phi^\iota_t \text{ is } \mathcal{F}_t \text{-measurable}
\]

\[
= \sum_{t \in T} E_{\pi'}[\Phi^\iota_t R_t].
\]

Then applying Theorem 2.1 or Corollary 2.1 permit then to conclude.

In the next, we will always suppose that there exists an asset \( R \) with positive returns and without shortselling constraints.

### 3.1 Complete and incomplete markets

As usually done in the FTAP, we will focus on the existence of an equivalent martingale measure and we do not work on it uniqueness.

First, we associate to the following \( N \) risky assets some cash-flows indexed by \( \mathcal{I} \). Let \( \iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I} \),

\[
\Phi^\iota_t(\omega) = -S^i_t(\omega) I_A(\omega) I_{\tau_1}(t, \omega) + S^i_t(\omega) I_A(\omega) I_{\tau_2}(t, \omega).
\]
The cash-flow $\Phi_i$ corresponds to the following strategy: buy one unit of asset $i$ at time $\tau_1$ if the event $A$ occurs and sell it at time $\tau_2$ always if $A$ occurs. Notice that $-\Phi^{(i,t,A,\tau_1,\tau_2)} = \Phi^{(i,t,A,\tau_2,\tau_1)}$. There is no shortselling constraints, and hence it is not necessary to specify if the buying date occurs before the selling date or not.

Notice that the implicit associated strategies are simple. We can include more complicated strategies, using linear combinations with positive coefficients of the preceding cash-flows. For example, if an investor wants to buy a share of asset $i$ at time $t_1$, sell a half at time $t_2$, and the other half at time $t_3$, he will choose a half unit of $\Phi^{i,t_1,\Omega,t_1,t_2}$ and a half unit of $\Phi^{i,t_1,\Omega,t_1,t_3}$.

We obtain then the classical result:

**Theorem 3.2** The securities price model is arbitrage free if and only if there exists a martingale measure $\pi'$ such that every renormalized price process is a martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$, that is for all $t \in T$,

$$\frac{S^i_t}{R_t} = E_{\pi'}[\frac{S^i_{t+1}}{R_{t+1}}/\mathcal{F}_t].$$

**Proof of Theorem 3.2:**

If we apply corollary 3.1 to $\Phi^{i,T,A,t,t+1}$ (and $\Phi^{i,t,A,t+1,t}$) for all $t \in T$ and for all $A \in \mathcal{F}_t$, we find that,

$$E_{\pi'}[(-\frac{S^i_t}{R_t} + \frac{S^i_{t+1}}{R_{t+1}})I_A] = 0.$$ 

Recalling that $\frac{S^i_t}{R_t}$ is $\mathcal{F}_t$-measurable and $A$ is a arbitrarily chosen in $\mathcal{F}_t$, we find that,

$$\frac{S^i_t}{R_t} = E_{\pi'}[\frac{S^i_{t+1}}{R_{t+1}}/\mathcal{F}_t].$$

Conversely, we prove that if $\frac{S^i_t}{R_t}$ is a $\pi'$-martingale then for all $t \in \mathcal{I}$,

$$\sum_{t \in \mathcal{T}} E_{\pi'}[\frac{\Phi^i_t}{R_t}] = 0,$$

which allows us to conclude from Corollary 3.1.

Let $t \in \mathcal{T}$,

$$\sum_{t \in \mathcal{T}} E_{\pi'}[\frac{\Phi^i_t}{R_t}] = \sum_{t \in \mathcal{T}} E_{\pi'}[\frac{-S^i_t I_A I_{\tau_1}(t,.) + S^i_t I_A I_{\tau_2}(t,.)}{R_t}],$$

$$= E_{\pi'}\left[\frac{-S^i_{\tau_1} I_A}{R_{\tau_1}} + \frac{S^i_{\tau_2} I_A}{R_{\tau_2}}\right],$$

from the definition of the conditional expectation and from $A \in \mathcal{F}_t$. 

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\[E_{\pi'}\left[-E_{\pi'}\left[ \frac{S_{\tau_1}}{R_{\tau_1}} / F_t \right] I_A \right] + E_{\pi'}\left[ E_{\pi'} \left[ \frac{S_{\tau_2}}{R_{\tau_2}} / F_t \right] I_A \right]
\]

recalling that \( \frac{S}{R} \) is a martingale under \( \pi' \),

\[E_{\pi'}\left[-\frac{S_t^i}{R_t} I_A + \frac{S_t^i}{R_t} I_A \right] = 0\]

In the context of Remark 3.1, \( \frac{R_t}{R_{t+1}} = \frac{1}{1+r_{t+1}} \) and we find that,

\[S_t^i = \frac{1}{1+r_{t+1}} E_{\pi'}[S_{t+1}^i / F_t].\]

3.2 Shortselling Constraints

In this context, the investor is not allowed to sell assets that she/he does not own. We obtain the following Theorem:

**Theorem 3.3** The securities price model is arbitrage free if and only if there exists an equivalent probability measure \( \pi' \) such that each renormalized price process is a supermartingale with respect to the filtration \( I_F \) and the probability measure \( \pi' \), that is for all \( t \in T \),

\[
\frac{S_t^i}{R_t} \geq E_{\pi'}[\frac{S_{t+1}^i}{R_{t+1}} / F_t].
\]

**Proof of Theorem 3.3:**

We use the same kind of representation for the \( N \) risky assets as in the previous section, but now the shortselling impossibility imposes that the investor must buy before he sells. Let \( \iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I} \), where \( \tau_1 \leq \tau_2 \),

\[
\Phi_{\iota}^i(\omega) = -S_t^i(\omega) I_A(\omega) I_{\tau_1}(t, \omega) + S_t^i(\omega) I_A(\omega) I_{\tau_2}(t, \omega).
\]

Applying Theorem 3.1 to \( \Phi_{i,t,A,t+1}^{i,t,A,t+1} \) (buy one unit of asset \( i \) at time \( t \) if the event \( A \in F_t \) occurs and sell \( i \) at \( t+1 \) if \( A \) occurs), for all \( t \in T \) and all \( A \in F_t \), we find that

\[E_{\pi'}\left[-\frac{S_t^i}{R_t} + \frac{S_{t+1}^i}{R_{t+1}} I_A \right] \leq 0.\]
Recalling that $\frac{S_t^i}{R_t}$ is $\mathcal{F}_t$-measurable and $A$ arbitrarily chosen in $\mathcal{F}_t$, we have that, $\frac{S_t^i}{R_t} \geq E_{\pi'} \left[ \frac{S_{t+1}^i}{R_{t+1}} / \mathcal{F}_t \right]$.

Conversely, let $\iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I}$, such that $\tau_1 \leq \tau_2$, then,

$$\sum_{t \in T} E_{\pi'} \left[ \Phi_t^i \right] = \sum_{t \in T} E_{\pi'} \left[ -\frac{S_t^i I_A I_{\tau_1}(t, \cdot) + S_t^i I_A I_{\tau_2}(t, \cdot)}{R_t} \right]$$

using the definition of the conditional expectation and recalling that $A \in \mathcal{F}_t$,

$$= E_{\pi'} \left[ -E_{\pi'} \left[ \frac{S_{\tau_1}^i}{R_{\tau_1}} / \mathcal{F}_t \right] I_A + E_{\pi'} \left[ \frac{S_{\tau_2}^i}{R_{\tau_2}} / \mathcal{F}_t \right] I_A \right]$$

using again the definition of the conditional expectation and $t \leq \tau_1 \leq \tau_2$,

$$= E_{\pi'} \left[ -E_{\pi'} \left[ \frac{S_{\tau_1}^i}{R_{\tau_1}} / \mathcal{F}_t \right] I_A + \left( -E_{\pi'} \left[ \frac{S_{\tau_2}^i}{R_{\tau_2}} / \mathcal{F}_t \right] I_A \right) \left( -E_{\pi'} \left[ \frac{S_{\tau_2}^i}{R_{\tau_2}} / \mathcal{F}_t \right] I_A \right) \right]$$

using the super martingale property of $\frac{S}{R}$,

$$\leq E_{\pi'} \left[ \left( -E_{\pi'} \left[ \frac{S_{\tau_1}^i}{R_{\tau_1}} / \mathcal{F}_t \right] I_A \right) \left( -E_{\pi'} \left[ \frac{S_{\tau_1}^i}{R_{\tau_1}} / \mathcal{F}_t \right] I_A \right) \right]$$

$$\leq 0$$

In the context of Remark 3.1, we find that,

$$\frac{S_t^i}{R_t} \geq \frac{1}{1 + \tau_{t+1}} E_{\pi'} \left[ \frac{S_{t+1}^i}{R_{t+1}} / \mathcal{F}_t \right] .$$

### 3.3 Shortselling costs

The investor can now sell assets before buying them, but this opportunity will be costly. We suppose that there exist two real-valued $\mathcal{F}$-adapted stochastic process, $S^i$ and $S'^i$. If the investor want to buy one unit of asset $i$ at time $t_1$, she/he will pay $S_{t_1}^i$, and if she/he sells it at $t_2$, she/he will receive $S_{t_2}^i$. On the other hand, if she/he sells short one unit of asset $i$ at time $t_1$, she/he will receive $S'^{i}_{t_1}$, and when she/he will buy it at $t_2$, she/he will pay $S'^{i}_{t_2}$. With this modelization the return of a long position $\frac{S_{t_1}^i}{S_{t_2}^i}$ is possibly different of the return of a short position $\frac{S_{t_2}^i}{S_{t_1}^i}$.

We obtain the following Theorem:
Theorem 3.4 The securities price model is arbitrage free if and only if there exists an equivalent probability measure $\pi'$ such that each renormalized price process $S^i$ (resp. $S'^i$) is a supermartingale (resp. submartingale) with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$, that is for all $t \in T$,

$$E_{\pi'}[\frac{S^i_{t+1}}{R_{t+1}}/\mathcal{F}_t] \leq \frac{S^i_t}{R_t}$$

and

$$E_{\pi'}[\frac{S'^i_{t+1}}{R_{t+1}}/\mathcal{F}_t].$$

Proof of Theorem 3.4

To take into account the shortselling costs, we consider two families of cash flows. Let $\iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I}$ such that $\tau_1 \leq \tau_2$,

$$\Phi_i^t(\omega) = -S^i_t(\omega)A(t, \omega) + S'^i_t(\omega)A(t, \omega).$$

$$\Phi'^i_t(\omega) = S'^i_t(\omega)A(t, \omega) - S^i_t(\omega)A(t, \omega).$$

Notice that if the cash flow $\Phi^i_{t_1:t_2}$ belongs to our portfolio, then between $t_1$ and $t_2$ and in the event $A$, we own the $i^{th}$ asset. Therefore, if we sell the asset $i$ at $t_3 \in [t_1, t_2]$ and then buy it at $t_4 \in [t_3, t_2]$, both in the event $A$, we will not sell it short, and we will not use the process $S'^i$. To take this case into account, just consider the cash flow $\Phi^i_{t_1:t_2}$ and $\Phi'^i_{t_3:t_4}$ (there is no transaction cost). Note that the choice of $\Phi^i_{t_1:t_2}$ and $\Phi'^i_{t_3:t_4}$ is also possible but since it is costly, the agents will not consider it.

The proof of Theorem 3.4 is similar as the one of Theorem 3.3.

3.4 Transaction costs

If we assume that there is a bid-ask spread, then we can model this situation by two real-valued $\mathcal{F}$-adapted stochastic process. We denote by $S^i$ the buying price process of the asset $i$ and by $S'^i$ its selling price. We obtain then the following Theorem :

Theorem 3.5 The securities price model is arbitrage free if and only if there exists an equivalent probability measure $\pi'$, and a process $S'^i$ between $S^i$ and $S'^i$, such that the renormalized process $S'^i$ is a martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$, that is for all $t \in T$,

$$\frac{S'^i_t}{R_t} = E_{\pi'}[\frac{S'^i_{t+1}}{R_{t+1}}/\mathcal{F}_t].$$

Proof of Theorem 3.5 :

We consider the following family of cash flows indexed by $\iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I}$ in order to represent the $N$ risky assets.

$$\Phi^i_t(\omega) = -S^i_t(\omega)A(t, \omega) + S'^i_t(\omega)A(t, \omega).$$
Notice that $-\Phi(i,t,A,\tau_1,\tau_2) = \Phi(i,t,A,\tau_2,\tau_1)$. There is no shortselling constraints and it is unnecessary to tell if we have bought or sold first.

Using Theorem 3.1 with $\Phi(i,t,A,\tau_1,\tau_2)$, we find that,

$$E_{\pi'}\left[ \frac{S^i_{\tau_2}}{R_{\tau_2}} / \mathcal{F}_t \right] \leq E_{\pi'}\left[ \frac{S^i_{\tau_1}}{R_{\tau_1}} / \mathcal{F}_t \right]. \quad (3.1)$$

Consider the two following processes,

$$\tilde{S}^i_t = \max_{\tau \in S_{t,T}} E_{\pi'}\left[ \frac{S^i_{\tau}}{R_{\tau}} / \mathcal{F}_t \right],$$

$$\hat{S}^i_t = \min_{\tau \in S_{t,T}} E_{\pi'}\left[ \frac{S^i_{\tau}}{R_{\tau}} / \mathcal{F}_t \right].$$

It is straightforward to see from the definition of $\tilde{S}^i$ and $\hat{S}^i$ that,

$$\tilde{S}^i \geq S^i$$

and

$$\hat{S}^i \leq S^i.$$

Let $\tau^*$ be an optimal stopping time of $(\frac{S^i_{\tau}}{R_{\tau}})_{k \in [t,T]}$ (the stopping times $\tau^*$ exists since the set $S_{t,T}$ is finite). By definition,

$$\frac{S^i_{\tau^*}}{R_{\tau^*}} = \max_{\tau \in S_{t,T}} E_{\pi'}\left[ \frac{S^i_{\tau}}{R_{\tau}} / \mathcal{F}_t \right] = \tilde{S}^i_t.$$

This leads to,

$$E_{\pi'}\left[ \frac{\tilde{S}^i_t}{R_t} / \mathcal{F}_{t-1} \right] = E_{\pi'}\left[ \frac{S^i_{\tau^*}}{R_{\tau^*}} / \mathcal{F}_{t-1} \right] \leq \max_{\tau \in S_{t-1,T}} E_{\pi'}\left[ \frac{S^i_{\tau}}{R_{\tau}} / \mathcal{F}_{t-1} \right] = \frac{\tilde{S}^i_{t-1}}{R_{t-1}},$$

where we have used that $\tau^* \in S_{t,T} \subset S_{t-1,T}$. The process $\frac{\tilde{S}^i}{R_t}$ is then a super-martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$. Using the same arguments, $\frac{\hat{S}^i}{R_t}$ is a sub-martingale with respect to $\mathcal{F}$ and $\pi'$.

If we consider the strategy that consists in buying and selling at the same stopping time $\tau \in S_{t,T}$, we find that $S^i_{\tau} \leq S^i$. This is quite intuitive: the selling price is less or equal than the buying price. Next, we will show that $S^i_{\tau} \leq \hat{S}^i$. Let $t \in [0,T]$, and $\tau_1, \tau_2 \in S_{t,T}$, recalling Equation (3.1), we have,

$$E_{\pi'}\left[ \frac{S^i_{\tau_2}}{R_{\tau_2}} / \mathcal{F}_t \right] \leq E_{\pi'}\left[ \frac{S^i_{\tau_1}}{R_{\tau_1}} / \mathcal{F}_t \right].$$
Taking the maximum on $\tau_2 \in S_{t,T}$ in the left-hand side of the previous equation, and then the minimum on $\tau_1 \in S_{t,T}$ in the right-hand side, we find immediately that $\tilde{S}_i^\prime \leq \tilde{S}_i^\prime$.

Finally, we have showed that there exists a process $\tilde{S}_i^\prime$ (resp. $\tilde{S}_i$) which is after renormalization a supermartingale (resp. submartingale), and such that,

$$S_i^\prime \leq \tilde{S}_i^\prime \leq \tilde{S}_i \leq S_i.$$

To end the proof, it remains to show that there exists a martingale $S_{ii}$ with respect to $\mathcal{F}$ and $\pi'$ such that $\tilde{S}_i \leq S_{ii} \leq \tilde{S}_i$. To do that, we use the following lemma, which proof is carried out later.

**Lemma 3.1** Let $(U_t)_{t \in T}$ be a supermartingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi$, and $(V_t)_{t \in T}$ be a submartingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi$, such that $U_t \leq V_t$ for all $t \in T$. Then, there exists a martingale $(M_t^*)_{t \in T}$ with respect to the filtration $\mathcal{F}$ and the probability measure $\pi$ such that $U_t \leq M_t^* \leq V_t$ for all $t \in T$.

Choosing $U_i = \tilde{S}_i / R$ and $V_i = \tilde{S}_i / R$, we find that there exists a martingale $M_i^*$ between $U_i$ and $V_i$ and we choose $S_{ii} = M_i^* R$.

Conversely if we have such a martingale, notice that,

$$\frac{S_{ii}}{R_t} = E_{\pi'} \left[ \frac{S_{ii}}{R_{\tau_1}} / \mathcal{F}_t \right] \leq E_{\pi'} \left[ \frac{S_{ii}}{R_{\tau_1}} / \mathcal{F}_t \right] \quad \text{and}$$

$$E_{\pi'} \left[ \frac{S_{ii}}{R_{\tau_1}} / \mathcal{F}_t \right] \leq E_{\pi'} \left[ \frac{S_{ii}}{R_{\tau_1}} / \mathcal{F}_t \right] = \frac{S_{ii}}{R_t}.$$

Consequently, let $i \in I$,

$$\sum_{t \in T} E_{\pi'} \left[ \frac{\Phi_t^i}{R_t} \right] = \sum_{t \in T} E_{\pi'} \left[ -\frac{S_{ii} I_A I_{\tau_1} (t,.) + S_{ii} I_A I_{\tau_2} (t,.)}{R_t} \right]$$

$$= E_{\pi'} \left[ -\frac{S_{ii} I_A}{R_{\tau_1}} + \frac{S_{ii} I_A}{R_{\tau_2}} \right]$$

using the definition of the conditional expectation and $A \in \mathcal{F}_t$,

$$= E_{\pi'} \left[ -E_{\pi'} \left[ \frac{S_{ii}}{R_{\tau_1}} I_A / \mathcal{F}_t \right] I_A \right] + E_{\pi'} \left[ E_{\pi'} \left[ \frac{S_{ii}}{R_{\tau_2}} / \mathcal{F}_t \right] I_A \right]$$

$$\leq E_{\pi'} \left[ -\frac{S_{ii} I_A}{R_t} + \frac{S_{ii} I_A}{R_t} \right]$$

$$\leq 0.$$
Proof of Lemma 3.1:

We will argue by induction. Using the supermartingale (resp. submartingale) property of the process $U$ (resp. $V$) and denoting by $E$ the expected value under $\pi$, we find that,

$$E[V_1/F_0] \geq V_0 \text{ and } E[U_1/F_0] \leq U_0.$$  

Let $M^\lambda = \lambda V_1 + (1 - \lambda)U_1$, with $0 \leq \lambda \leq 1$.

It is straightforward to see that $E[M_1^\lambda/F_0]$ runs throughout the stochastic interval $[E[U_1/F_0], E[V_1/F_0]]$. An element of a closed, bounded interval is a convex combination of its extremal points, then there exists $\lambda^*_1$ such that

$$U_0 \leq E[M_1^{\lambda^*_1}/F_0] \leq V_0. \tag{3.2}$$

We denote by $[\lambda_{1\text{min}}, \lambda_{1\text{max}}]$ the range of $\lambda^*_1$ such that Equation (3.2) is satisfied. Then,

$$U_1 \leq M_1^{\lambda_{1\text{min}}} \leq M_1^{\lambda_{1\text{max}}} \leq V_1.$$  

At the second step, we find that,

$$E[V_2/F_1] \geq V_1 \geq M_1^{\lambda_{1\text{max}}} \text{ and } E[U_2/F_1] \leq U_1 \leq M_1^{\lambda_{1\text{min}}}.$$  

Let $\lambda$ be a $F_1$-measurable random variable taking value in $[0, 1]$, we consider the following process,

$$M_2^\lambda = \lambda V_2 + (1 - \lambda)U_2.$$  

Notice that $E[M_2^\lambda/F_1]$ runs throughout the stochastic interval $[E[U_2/F_1], E[V_2/F_1]]$. Now the stochastic interval $[M_1^{\lambda_{1\text{min}}}, M_1^{\lambda_{1\text{max}}}]$ is a subset of $[U_1, V_1]$, which is itself included in $[E[U_2/F_1], E[V_2/F_1]]$, so there exists a $F_1$ measurable random variable $\lambda^*_2$ (it is built at every node of date 1, only from the informations of date 1) such that

$$U_1 \leq M_1^{\lambda_{1\text{min}}} \leq E[M_2^{\lambda^*_2}/F_1] \leq M_1^{\lambda_{1\text{max}}} \leq V_1. \tag{3.3}$$

As before, we denote $[\lambda_{2\text{min}}, \lambda_{2\text{max}}]$ the stochastic interval of $F_1$ measurable random variable $\lambda^*_2$ such that Equation (3.3) is satisfied.
We construct by induction pairs of random variables respectively $\mathcal{F}_2, \ldots, \mathcal{F}_{T-1}$ meausurables, $(\lambda_2^{\min}, \lambda_2^{\max}), \ldots, (\lambda_T^{\min}, \lambda_T^{\max})$ such that, for all $p \in \{0, \ldots, T-1\}$ and $\lambda \in \left[\lambda_p^{\min}, \lambda_p^{\max}\right]$,

$$U_p \leq M_p^{\lambda_{p+1}} \leq E[M_{p+1}^{\lambda}/\mathcal{F}_p] \leq M_{p}^{\lambda_{p+1}} \leq V_p.$$ 

Now, we choose a $\mathcal{F}_{T-1}$-measurable random variable $\lambda_T^* \in \left[\lambda_T^{\min}, \lambda_T^{\max}\right]$, and we denote by

$$M_T^* = \lambda_T V_T + (1 - \lambda_T)U_T,$$

we have then that

$$U_{T-1} \leq M_{T-1}^{\lambda_{T-1}^{\min}} \leq E[M_T^{\lambda}/\mathcal{F}_{T-1}] \leq M_{T-1}^{\lambda_{T-1}^{\max}} \leq V_{T-1}.$$ 

If we denote

$$M_{T-1}^* = E[M_T^*/\mathcal{F}_{T-1}],$$

Then, there exists an $\mathcal{F}_{T-2}$-measurable random variable $\lambda_{T-1} \in \left[\lambda_{T-1}^{\min}, \lambda_{T-1}^{\max}\right]$, such that

$$M_{T-1}^* = \lambda_{T-1} V_{T-1} + (1 - \lambda_{T-1})U_{T-1}.$$ 

Iterating the same reasoning, we construct step after step a martingale $M^*$ with respect to $\mathcal{F}$ and $\pi'$, such that $U \leq M^* \leq V$. 

### 3.5 The dividends case

Suppose that every asset $i$ pays dividends. We suppose that every asset pays dividends (possibly equal to zero) from the date following its buying date up to its selling date. Conversely, an investor buying this asset will pay dividends from the date following its buying date up to its buying date. The value $D_i^j(\omega)$ is the dividend paid by asset $i$ in state $\omega$ and at time $t$. We suppose that the process $D_i^j$ is $\mathcal{F}$-adapted.

#### 3.5.1 Complete and incomplete markets

In the complete and incomplete markets case, our main theorem states as follows:

**Theorem 3.6** The securities price model is arbitrage free if and only if there exists an equivalent probability measure $\pi'$ such that for all $i$, that is
for all $t \in T$, $S_t^i/T_t = E_{\pi'}[S_{t+1}^i+D_{t+1}^i/R_{t+1}/F_t]$. Equivalently, for all $t \in T$,

$$S_t^i/R_t + \sum_{u=0}^{t} D_u^i/R_u = E_{\pi'}\left[\frac{S_{t+1}^i+D_{t+1}^i}{R_{t+1}}\right],$$

and the process $S_t^i/R_t + \sum_{u=0}^{t} D_u^i/R_u$ is a martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$.

**Proof:**

First, we define the associated family of cash-flows. Let $\iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I}$,

$$\Phi_t^i(\omega) = -S_t^i(\omega)I_A(\omega)I_{\tau_1}(t, \omega) + S_t^i(\omega)I_A(\omega)I_{\tau_2}(t, \omega) + D_t^i(\omega)I_{\tau_1 < \tau_2}(\omega)I_{[\tau_1, \tau_2]}(t, \omega)I_A(\omega) - D_t^i(\omega)I_{\tau_2 < \tau_1}(\omega)I_{[\tau_1, \tau_2]}(t, \omega)I_A(\omega).$$

With the notations, $I_{\tau_1 < \tau_2}(\omega) = 1$ if $\tau_1(\omega) < \tau_2(\omega)$ and zero else, and $I_{[\tau_1, \tau_2]}(t, \omega) = 1$ if $t \in [\tau_1(\omega), \tau_2(\omega)]$.

This means that it is possible to buy one unit of asset $i$ if the event $A$ occurs, at the random time $\tau_1$, and to sell it back at the random time $\tau_2$ always if the event $A$ occurs. If the buying date $\tau_1(\omega)$ predecease the buying date $\tau_2(\omega)$, the investor receive dividends between $\tau_1(\omega)$ and $\tau_2(\omega)$, else she/he must pay dividends between $\tau_2(\omega)$ and $\tau_1(\omega)$. Notice that $-\Phi_t^i(t, A, \tau_1, \tau_2) = \Phi_t^{(i, t, A, \tau_1, \tau_2)}$, because there is no short-selling constraints.

Now apply Corollary 3.1 to $\Phi_t^{i, t, A, t, t+1}$ (buy at $t$ one unit of $i$ if the event $A \in \mathcal{F}_t$ occurs and sell it at $t + 1$ always if $A$ occurs), we get that,

$$S_t^i/R_t = E_{\pi'}\left[\frac{S_{t+1}^i+D_{t+1}^i}{R_{t+1}}\right].$$

Recalling that $\sum_{u=0}^{t} D_u^i/R_u$ is $\mathcal{F}_t$-measurable,

$$S_t^i/R_t + \sum_{u=0}^{t} D_u^i/R_u = E_{\pi'}\left[\frac{S_{t+1}^i+D_{t+1}^i}{R_{t+1}}\right] + \sum_{u=0}^{t+1} D_u^i/R_u.\]$$

Notice that using the law of the iterated expectations,

$$S_0^i = E_{\pi'}\left[\frac{S_T^i}{R_T} + \sum_{t=1}^{T} D_t^i/R_t\right].$$
Conversely, let \( \iota \in I \), we have that,

\[
\sum_{t \in T} E_{\pi'} \left[ \Phi^i_t \right] = \sum_{t \in T} E_{\pi'} \left[ -S_i I_{t_1}(t,.) + S_i I_{t_2}(t,.) + D_i I_{t_1 < t_2 I_{t_1,t_2}(t,.)} I_A \right] \\
- \frac{D_i I_{t_2 < t_1 I_{t_2,t_1}(t,.)}}{R_t} I_A \\
= E_{\pi'} \left[ \left( -\frac{S_i}{R_{t_1}} + \frac{S_i}{R_{t_2}} \right) I_A \right] + E_{\pi'} \left[ \left( \sum_{u=t_1+1}^{t_2} \frac{D_i^u}{R_u} I_{t_1 < t_2} - \sum_{u=t_2+1}^{t_1} \frac{D_i^u}{R_u} I_{t_2 < t_1} \right) I_A \right]
\]

If we remark that,

\[
\sum_{u=t_1+1}^{t_2} \frac{D_i^u}{R_u} I_{t_1 < t_2} - \sum_{u=t_2+1}^{t_1} \frac{D_i^u}{R_u} I_{t_2 < t_1} = \\
\left( \sum_{u=0}^{t_2} \frac{D_i^u}{R_u} - \sum_{u=0}^{t_1} \frac{D_i^u}{R_u} \right) I_{t_1 < t_2} - \left( \sum_{u=0}^{t_1} \frac{D_i^u}{R_u} - \sum_{u=0}^{t_2} \frac{D_i^u}{R_u} \right) I_{t_2 < t_1} = \sum_{u=0}^{t_2} \frac{D_i^u}{R_u} - \sum_{u=0}^{t_1} \frac{D_i^u}{R_u}.
\]

Then, we get that,

\[
\sum_{t \in T} E_{\pi'} \left[ \Phi^i_t \right] = E_{\pi'} \left[ \left( -\frac{S_i}{R_{t_1}} + \sum_{u=0}^{t_1} \frac{D_i^u}{R_u} + \frac{S_i}{R_{t_2}} + \sum_{u=0}^{t_2} \frac{D_i^u}{R_u} \right) I_A \right] \\
\text{using the definition of the expected value and noting that } A \in F_t, \\
= E_{\pi'} \left[ \left( -E_{\pi'} \left[ \frac{S_i}{R_{t_1}} + \sum_{u=0}^{t_1} \frac{D_i^u}{R_u} / F_t \right] + E_{\pi'} \left[ \frac{S_i}{R_{t_2}} + \sum_{u=0}^{t_2} \frac{D_i^u}{R_u} / F_t \right] \right) I_A \right] \\
\text{using the martingale property of } \frac{S_i}{R_t} + \sum_{u=0}^{t} \frac{D_i^u}{R_u}, \\
= E_{\pi'} \left[ \left( -\frac{S_i}{R_t} + \sum_{u=0}^{t} \frac{D_i^u}{R_u} \right) + \frac{S_i}{R_t} + \sum_{u=0}^{t} \frac{D_i^u}{R_u} \right] I_A \\
= 0
\]

\[\boxed{}\]

### 3.5.2 Shortselling costs

There exists two \( \mathcal{F} \)-adapted stochastic process, \( S^i \) and \( S'^i \) as in section 3.2. Using the same line of argument as before, we find that,
Theorem 3.7 The securities price model is arbitrage free if and only if there exists an equivalent probability measure $\pi'$ such that for all $i$, that is for all $t \in T$,

$$E_{\pi'}\left[ \frac{S_{i+1}^t + D_{i+1}^t}{R_{i+1}} / \mathcal{F}_t \right] \leq \frac{S_i^t}{R_t} \text{ and } \frac{S_i^t}{R_t} \leq E_{\pi'}\left[ \frac{S_{i+1}^t + D_{i+1}^t}{R_{i+1}} / \mathcal{F}_t \right].$$

3.6 Proportional transaction costs

As in section 3.3, we model the transaction costs with two $\mathcal{F}$-adapted stochastic process, $S^i$ (buying price) and $S'^i$ (selling price). In the transaction costs framework, the theorem states as follows,

Theorem 3.8 The securities price model is arbitrage free if and only if there exists an equivalent probability measure $\pi'$ and a process $S'^i$, which lies between $S'^i$ and $S^i$, and such that the process $(\sum_{u=0}^t D_u^i / R_u)_{t \in T}$ is a martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$.

Proof:
Let $i = (i, t, A, \tau_1, \tau_2) \in I$,

$$\Phi^i_t(\omega) = -S^i(t, \omega)I_{\tau_1 < \tau_2} + S'^i(t, \omega)I_{\tau_1 < \tau_2} - D^i(t, \omega)(I_{\tau_1 < \tau_2} - I_{\tau_2 < \tau_1}).$$

In this case, shortselling is allowed and the cash-flows $-\Phi^i(t, A, \tau_1, \tau_2) = \Phi^i(t, A, \tau_2, \tau_1)$ is feasible.

Applying Theorem 3.1 to $\Phi^i(t, A, \tau_1, \tau_2)$, we find that,

$$E_{\pi'}\left[ \frac{S'^i_{\tau_2}}{R_{\tau_2}} + \sum_{u=\tau_1+1}^{\tau_2} \frac{D_u^i}{R_u} I_{\tau_1 < \tau_2} / \mathcal{F}_t \right] \leq E_{\pi'}\left[ \frac{S_i^t_{\tau_1}}{R_{\tau_1}} + \sum_{u=\tau_2+1}^{\tau_1} \frac{D_u^i}{R_u} I_{\tau_2 < \tau_1} / \mathcal{F}_t \right].$$

If we recall that,

$$\sum_{u=\tau_1+1}^{\tau_2} \frac{D_u^i}{R_u} I_{\tau_1 < \tau_2} = \sum_{u=0}^{\tau_1} \frac{D_u^i}{R_u} - \sum_{u=0}^{\tau_2} \frac{D_u^i}{R_u}.$$

Then, this leads to,

$$E_{\pi'}\left[ \frac{S'^i_{\tau_2}}{R_{\tau_2}} + \sum_{u=0}^{\tau_2} \frac{D_u^i}{R_u} / \mathcal{F}_t \right] \leq E_{\pi'}\left[ \frac{S_i^t_{\tau_1}}{R_{\tau_1}} + \sum_{u=0}^{\tau_1} \frac{D_u^i}{R_u} / \mathcal{F}_t \right].$$

(3.4)
Let us define $\tilde{S}$ and $\tilde{S}'$ by,

$$\frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} = \max_{\tau \in \mathcal{S}_{t,T}} E_{\pi'} \left[ \frac{S^i_{\tau}}{R_{\tau}} + \sum_{u=0}^{\tau} \frac{D^i_u}{R_u} / \mathcal{F}_t \right],$$

$$\frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} = \min_{\tau \in \mathcal{S}_{t,T}} E_{\pi'} \left[ \frac{S^i_{\tau}}{R_{\tau}} + \sum_{u=0}^{\tau} \frac{D^i_u}{R_u} / \mathcal{F}_t \right].$$

We get that,

$$\frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \geq \frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \quad \text{and} \quad \frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \leq \frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}.$$

Let $\tau^*$ be the stopping time of the process $(\frac{S^i_t}{R_t} + \sum_{u=0}^{k} \frac{D^i_u}{R_u})_{k \in [t,T]}$ (this stopping time $\tau^*$ exists because $S_{t,T}$ is finite). Then,

$$\frac{S^i_{\tau^*}}{R_{\tau^*}} + \sum_{u=0}^{\tau^*} \frac{D^i_u}{R_u} = \max_{\tau \in \mathcal{S}_{t,T}} E_{\pi'} \left[ \frac{S^i_{\tau}}{R_{\tau}} + \sum_{u=0}^{\tau} \frac{D^i_u}{R_u} / \mathcal{F}_t \right] = \frac{\tilde{S}^i_{t}}{R_{t}} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}.$$

This shows that,

$$E_{\pi'} \left[ \frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} / \mathcal{F}_{t-1} \right] = E_{\pi'} \left[ \frac{S^i_{\tau^*}}{R_{\tau^*}} + \sum_{u=0}^{\tau^*} \frac{D^i_u}{R_u} / \mathcal{F}_{t-1} \right] \leq \frac{\tilde{S}^i_{t}}{R_{t-1}} + \sum_{u=0}^{t-1} \frac{D^i_u}{R_u},$$

where we have notice that $\tau^* \in \mathcal{S}_{t,T} \subset \mathcal{S}_{t-1,T}$. The process $\frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}$ is a supermartingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$. Using the same arguments, $\frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}$ is then a supermartingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$.

Taking the maximum on $\tau_2 \in \mathcal{S}_{t,T}$ in the left-handside of Equation (3.4) and then the minimum on $\tau_1 \in \mathcal{S}_{t,T}$ in the right-handside, it follows that,

$$\frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \leq \frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}.$$

The process $\frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}$ (resp. $\frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}$) is a supermartingale (resp. submartingale), and,

$$\frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \leq \frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \leq \frac{S^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u} \leq \frac{\tilde{S}^i_t}{R_t} + \sum_{u=0}^{t} \frac{D^i_u}{R_u}.$$
Using Lemma 3.1, we find that there exists a martingale \( Z^{*i} \) with respect to the filtration \( \mathcal{F} \) and the probability measure \( \pi' \), such that \( Z^{*i}_t \) lies between 
\[
\frac{S^{i}_t}{R_t} + \sum_{u=0}^{t} \frac{D^{i}_u}{R_u} \quad \text{and} \quad \frac{S^{i}_t}{R_t} + \sum_{u=0}^{t} \frac{D^{i}_u}{R_u}.
\]
Denoting by \( S^{*i} \) the process
\[
S^{*i}_t := \left[ Z^{*i}_t - \sum_{u=0}^{t} \frac{D^{i}_u}{R_u} \right] R_t,
\]
we have proved the implication of the required result.

Conversely, let \( \iota = (i, t, A, \tau_1, \tau_2) \in \mathcal{I} \), define first the process \( Z^{*i} \) by,
\[
Z^{*i}_t := \frac{S^{i}_t}{R_t} + \sum_{u=0}^{t} \frac{D^{i}_u}{R_u}.
\]
The process \( Z^{*i} \) is a martingale with respect to the filtration \( \mathcal{F} \) and the probability measure \( \pi' \), and we have that,
\[
E_{\pi'} \left[ Z^{*i}_t / \mathcal{F}_t \right] \leq E_{\pi'} \left[ \frac{S^{i}_{\tau_1}}{R_{\tau_1}} + \sum_{u=0}^{\tau_1} \frac{D^{i}_u}{R_u} / \mathcal{F}_t \right]
\]
and
\[
E_{\pi'} \left[ \frac{S^{i}_{\tau_1}}{R_{\tau_1}} + \sum_{u=0}^{\tau_1} \frac{D^{i}_u}{R_u} / \mathcal{F}_t \right] \leq E_{\pi'} \left[ Z^{*i}_{\tau_1} / \mathcal{F}_t \right] = Z^{*i}_t.
\]

As previously done, for all \( t \in \mathcal{I} \),
\[
\sum_{t \in \mathcal{I}} E_{\pi'} \left[ \Phi^i_t / R_t \right] = \sum_{t \in \mathcal{I}} E_{\pi'} \left[ -\frac{S^{i}_t I_A I_{\tau_1}(t, .) + S^{i}_t I_A I_{\tau_2}(t, .)}{R_t} \right]
\]
\[
= E_{\pi'} \left[ -\frac{S^{i}_{\tau_1} I_A}{R_{\tau_1}} + \frac{S^{i}_{\tau_2} I_A}{R_{\tau_2}} \right]
\]
\[
= E_{\pi'} \left[ -E_{\pi'} \left[ \frac{S^{i}_{\tau_1}}{R_{\tau_1}} / \mathcal{F}_t \right] I_A \right] + E_{\pi'} \left[ E_{\pi'} \left[ \frac{S^{i}_{\tau_2}}{R_{\tau_2}} / \mathcal{F}_t \right] I_A \right]
\]
\[
\leq E_{\pi'} \left[ -\frac{S^{*i}_t I_A + S^{*i}_t I_A}{R_t} \right]
\]
\[
\leq 0
\]
4 Taxes

We focus now on the case of an investor paying taxes on capital gains and receiving tax credits for capital losses. We suppose that those tax payments and tax credits occur immediately, that is when the investor actually realises those gains and losses by trading shares of stock and not at the end of the year.

In order to justify our assumptions, we recall some rules of the French tax code. We refer to Lamorlette-Lamorlette (1995).

First, we justify the assumption of credit for capital losses. For an individual, this assumption is not true for a loss on real estate and personal estate. The capital loss can neither be credited on the capital gain from same kind of assets nor from other assets. It must be treat as an unproductive asset. However, for transferable securities, the capital losses can be deducted against capital gains on the same kind of assets, in the same year or in the next five years. This gains could also be deducted against capital gains of the MONEP and the MATIF. For the industrial and commercial gains, the capital losses can also be deducted against capital gains. Of course in those cases, this is not exactly a credit in case of loss, but this justify the assumption of compensation in case of loss.

Next, we present the different methods used to compute the capital gain. The first one consists in making the difference between the selling price and the buying price. Those two prices are face values. This is the most used method. Another one consists in making the difference between the selling price and the discounted buying price. This discounting takes into account the erosion by inflation or by use. For example, this method is used in the case of real estate and personal estate. The first method is a particular case of the second one with an erosion coefficient equal to one. In the next, we will use this formalization. More precisely, the erosion coefficient is modeled by an adapted and positive process $U$. If the investor buys one unit of asset $i$ at time $t_1$, and if sells it at time $t_2$, the capital gain or loss will be equal to $S_{t_2}^i - rac{U_{t_2}}{U_{t_1}} S_{t_1}^i$. The tax or the credit consists in $k$ percent of the capital gain or loss. Here the strong assumption is to use the same coefficient $k$ for the tax and the credit. Recall that the tax code imposes for some classes of investors, managing rules like First.In.First.Out or Last.In.Last.Out. We will not use this rules. To justify our position, we observe that the market contains many close substitutes assets so that investors can buy another security with similar risk and tax characteristic without violating the F.I.F.O or L.I.F.O rule (see Cadenillas-Pliska (1996), Constantinides (1983) and Dybvig-Koo (1995)).
Our main theorem states as follows:

**Theorem 4.1** If the market is arbitrage free then there exists an equivalent probability measure $\pi'$ such that $S^i$, discounted by a process $C$, is a martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$. This means that for all $i$, that is for all $t, t' \in T$ such that $t' \geq t$,

$$S^i_tC_t = E_{\pi'}[S^i_{t'}C_{t'}/\mathcal{F}_t], \text{ with } C_t := \frac{1}{R_t} - \frac{k}{U_t}E_{\pi'}[\frac{U_T}{R_T}/\mathcal{F}_t].$$

This means that, $S^i$ is a martingale via an actualization by the non risky asset corrected by the expected ratio, taxed at $k$, of the erosion by the non risky asset.

In the case of an erosion process $U$ equal to $R$, we find that, $C_t = \frac{1}{R_t}(1 - k)$.

**Corollary 4.1** If $U = R$ and if the market is arbitrage free then there exists an equivalent martingale measure $\pi'$ such that $S^i/R$ is a martingale with respect to the filtration $\mathcal{F}$ and the probability measure $\pi'$.

**Proof of Theorem 4.1** :

We first represent the $N$ risky assets through their cash-flows. Recall that we assume that an investor pays taxes on capital gains and receives credits losses. Let $\iota = (i, t, A, \tau_1, \tau_2) \in I$,

$$\Phi^i_t(\omega) = -S^i_t(\omega)I_A(\omega)I_{\tau_1}(t, \omega) + S^i_t(\omega)I_A(\omega)I_{\tau_2}(t, \omega)$$

$$- k \left( S^i_t(\omega) - \frac{U_t}{U_{\tau_1}}(\omega)S^i_{\tau_1}(\omega) \right) I_A(\omega)I_{\tau_2}(t, \omega)I_{\tau_1 < \tau_2}$$

$$+ k \left( S^i_t(\omega) - \frac{U_t}{U_{\tau_2}}(\omega)S^i_{\tau_2}(\omega) \right) I_A(\omega)I_{\tau_2}(t, \omega)I_{\tau_2 < \tau_1}.$$

Since short-selling is allowed, then $-\Phi^{(i,t,A,\tau_1,\tau_2)}(\omega) = \Phi^{(i,t,A,\tau_1,\tau_1)}(\omega)$ is an available cash-flow.

If we apply corollary 3.1 to $\Phi^{i,t_1,A,t_1,t_2,A} (\text{and } \Phi^{i,t_1,A,t_2,t_1})$, for all $t_1, t_2 \in T$, with $t_1 \leq t_2$, and for all $A \in \mathcal{F}_{t_1}$, we find that,

$$E_{\pi'}\left[ -\frac{S^i_{t_1}}{R_{t_1}} + \frac{S^i_{t_2} - k(S^i_{t_2} - \frac{U_{t_2}}{U_{t_1}}S^i_{t_1})}{R_{t_2}} I_A \right] = 0.$$
As \( \frac{S^i_{t_1}}{R_{t_1}} \) is \( \mathcal{F}_{t_1} \)-measurable, we find that,

\[
\frac{S^i_{t_1}}{R_{t_1}} = E_{\pi'} \left[ \frac{S^i_{t_2}}{R_{t_2}} - \frac{k}{R_{t_2}} \left( S^i_{t_2} - \frac{U_T}{U_t} S^i_{t_1} \right) / \mathcal{F}_{t_1} \right]. \tag{4.5}
\]

If we apply this equation between \( t \) and \( T \), then between \( t \) and \( t' \), and finally between \( t' \) and \( T \), with \( t \leq t' \leq T \), we find that,

\[
\frac{S^i_t}{R_t} = E_{\pi'} \left[ \frac{S^i_T}{R_T} - \frac{k}{R_T} \left( S^i_T - \frac{U_T}{U_t} S^i_t \right) / \mathcal{F}_t \right], \tag{4.6}
\]

\[
\frac{S^i_t}{R_t} = E_{\pi'} \left[ \frac{S^i_{t'}}{R_{t'}} - \frac{k}{R_{t'}} \left( S^i_{t'} - \frac{U_{t'}}{U_t} S^i_t \right) / \mathcal{F}_t \right], \tag{4.7}
\]

\[
\frac{S^i_{t'}}{R_{t'}} = E_{\pi'} \left[ \frac{S^i_T}{R_T} - \frac{k}{R_T} \left( S^i_T - \frac{U_T}{U_{t'}} S^i_{t'} \right) / \mathcal{F}_{t'} \right]. \tag{4.8}
\]

Equations (4.7) and (4.8) lead to,

\[
\frac{S^i_t}{R_t} = E_{\pi'} \left[ \frac{S^i_{t'}}{R_{t'}} - \frac{k}{R_{t'}} \left( S^i_{t'} - \frac{U_{t'}}{U_t} S^i_t \right) / \mathcal{F}_{t'} \right].
\]

Comparing with Equation (4.6), we get that,

\[
E_{\pi'} \left[ \frac{k}{R_T} \left( \frac{U_T}{U_{t'}} S^i_{t'} - \frac{U_T}{U_t} S^i_t \right) / \mathcal{F}_t \right] = E_{\pi'} \left[ \frac{k}{R_{t'}} \left( S^i_{t'} - \frac{U_{t'}}{U_t} S^i_t \right) / \mathcal{F}_{t'} \right].
\]

Considering Equation (4.7) again, we obtain that,

\[
\frac{S^i_t}{R_t} = E_{\pi'} \left[ \frac{S^i_{t'}}{R_{t'}} - \frac{k}{R_{t'}} \left( \frac{U_T}{U_{t'}} S^i_{t'} - \frac{U_T}{U_t} S^i_t \right) / \mathcal{F}_{t'} \right].
\]

As \( \frac{S^i_{t'}}{R_{t'}} \) is \( \mathcal{F}_{t'} \)-measurable, we find that,

\[
\frac{S^i_{t'}}{R_{t'}} = E_{\pi'} \left[ \frac{U_T}{R_T} / \mathcal{F}_{t'} \right] = E_{\pi'} \left[ \frac{S^i_{t'}}{R_{t'}} \left( \frac{1}{R_t} - \frac{k}{U_t} \left( \frac{U_T}{R_T} / \mathcal{F}_t \right) \right) / \mathcal{F}_{t'} \right].
\]

Now, if we denote by

\[
C_t = \frac{1}{R_t} - \frac{k}{U_t} E_{\pi'} \left[ \frac{U_T}{R_T} / \mathcal{F}_t \right],
\]

then we obtain that,

\[
S^i_tC_t = E_{\pi'} \left[ \frac{S^i_tC_{t'}}{\mathcal{F}_t} \right].
\]
References


