No-arbitrage in discrete-time under portfolio constraints

Abstract

In frictionless securities markets, the characterization of the no arbitrage condition by the existence of equivalent martingale measures in discrete time is known as the Fundamental Theorem of Asset Pricing. In the presence of convex constraints on the trading strategies, we extend this theorem under a closedness condition and a nondegeneracy assumption. We then provide connections with the super-replication problem solved in Föllmer and Kramkov (1997).

Keywords: Fundamental theorem of asset pricing, arbitrage, super-replication cost, convex portfolio constraints, stochastic process.

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1 Introduction

We study a discrete-time financial market consisting of $d$ risky assets, with the discounted price process, denoted $S$, and one riskless bond. In contrast with usual frictionless models, we consider the case where trading strategies are subject to portfolio constraints as in the framework of Cvitanić and Karatzas (1992,1993) and Karatzas and Kou (1996); see also the textbook exposition by Pliska (1997). Namely, given two convex sets $C^+$ and $C^-$ containing the origin, the vector of wealth proportions invested in the risky assets is constrained to lie in $C^+$ or $C^-$ depending on the sign of the current wealth process. A precise description of the model is provided in Section 2.

In this paper, we address the characterization of no arbitrage in such a financial market. Loosely speaking, an arbitrage opportunity is a way to produce nonnegative wealth with positive expected value out of nothing. Therefore, there should not exist such strategies in real markets. This problem has been studied in frictionless markets by Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981), Dalang, Morton and Willinger (1990), Schachermayer (1992) and Kabanov and Kramkov (1994). The Fundamental Theorem of Asset Pricing (FTAP) in this frictionless framework is the following: There is no arbitrage opportunity if and only if there exists an equivalent probability measure which turns the process $S$ into a martingale. Jouini and Kallal (1995) and Schürger (1996) provided an extension of the FTAP to the case where trading strategies are subject to the no short sales condition. The case of (closed) cone constraints $C$ on the amounts invested in the risky assets has been studied by Pham and Touzi (1999). Under a nondegeneracy assumption on the process $S$, they proved the following extension of the FTAP: There is no arbitrage opportunity if and only if there exists an equivalent probability measure $Q$ (with bounded density) such that $E^Q[\text{diag}(S_t)^{-1}(S_{t+1} - S_t)|\mathcal{F}_t] \in C^\circ$, for all $t$, where $C^\circ$ is the polar cone of $C$ and $\text{diag}(S_t)$ is the diagonal matrix whose $i$-th diagonal element is $S_{ti}$. This result has been extended by Brannath (1997) to the case of closed convex constraints on the invested amount.

In our constrained proportion-portfolio framework, the self-financed wealth process is written in a multiplicative form (see relation (2.2)). Therefore, whenever the initial capital is
zero, the wealth process is zero for any proportion-portfolio choice. We introduce here a more general definition of arbitrage. Given an initial wealth $x$ and a constrained portfolio strategy $\pi$, we define the excess wealth as the difference between the terminal wealth associated with $(x, \pi)$, and the initial wealth $x$. We then define an arbitrage opportunity as a way to produce a nonnegative excess wealth with positive expected value.

The main result of the paper is contained in Section 3 and proved in section 6. We show that there is no arbitrage if and only if there exists a pair of equivalent probability measures $(Q^+, Q^-)$ such that $E_{Q^+} [\text{diag}(S_t)^{-1}(S_{t+1} - S_t) | \mathcal{F}_t] \in \mathcal{C}^+$ and $E_{Q^-} [\text{diag}(S_t)^{-1}(S_{t+1} - S_t) | \mathcal{F}_t] \in -\mathcal{C}^-$, where $\mathcal{C} = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 0 \text{ for all } y \in C \}$. The last result is established for convex sets $C^+$ and $C^-$ with closed generated cones and under a nondegeneracy assumption as in Pham and Touzi (1999). Section 4 contains relevant examples of application of the main result.

In Section 5, we focus on the super-replication problem, i.e. the minimal initial wealth needed to hedge without risk some given contingent claim. Föllmer and Kramkov (1997) provided a dual formulation of the super-replication cost. We relate their assumption to the no-arbitrage concept, as in the usual literature on this problem.

## 2 The model

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t, t = 0, ..., T\}$ where $T > 0$ is a finite time horizon. The set $\mathcal{F}_t$ represents the whole information available at time $t$. We make the usual assumption that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

We first introduce some notation. The space $L_{t}^{0,d}$ is the space of all $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable functions. The space $L_{t}^{p,1}$ will be simply denoted $L_{t}^p$. For $1 \leq p \leq \infty$, the space $L_{t}^{p,d}$ is the Banach space of all $\mathbb{R}^d$-valued $\mathcal{F}_t$ measurable functions with finite $L^p$ norm. As before the superscript $d$ will be omitted when $d = 1$. We also use the classical notation: $\mathbb{R}^*_+ = \mathbb{R} \setminus \{0\}$, $L_{t}^{p,d}_+ = \{z \in L_{t}^{p,d} : z \geq 0 \text{ P a.s } \}$, for $0 \leq p \leq \infty$, and $|x| = \sum_{i=1}^{d} |x_i|$, for $x \in \mathbb{R}^d$.

The financial market model consists of one riskless asset with price process normalized to one and $d$ risky assets with price process $S = \{S_t = (S^1_t, ..., S^d_t)^*, t = 0, ..., T\}$ valued in
\((0, \infty)^d\) and \(\mathcal{F}\)-adapted. Here the notation \(\ast\) is for the transposition. We denote the return process associated with \(S\) by \(\{R_t = \text{diag}(S_{t-1})^{-1}(S_t - S_{t-1}), t = 1, \ldots, T\}\).

A trading strategy is a \(\mathbb{R}^d\)-valued \(\mathcal{F}\)-adapted process \(\pi = \{\pi_t = (\pi^1_t, \ldots, \pi^d_t)^\ast, t = 0, \ldots, T - 1\}\), where \(\pi^i_t\) represents the proportion of wealth invested in the \(i\)-th risky asset at time \(t\).

Given an initial wealth \(x \in \mathbb{R}^d\) and a trading strategy \(\pi\), it follows from the self-financing condition that the wealth process \(X^{x, \pi}\) is governed by:

\[
X_0^{x, \pi} = x \quad X_{t+1}^{x, \pi} = X_t^{x, \pi} (1 + \pi^* R_{t+1}), \quad \text{for } t = 0, \ldots, T - 1
\]

(2.1)

The induction equation (2.1) leads to

\[
X_t^{x, \pi} = x + \sum_{u=0}^{t-1} X_u^{x, \pi} \pi_u^* R_{u+1}, \quad \text{for } t = 0, \ldots, T.
\]

(2.2)

Remark 2.1 Our class of trading strategies based on proportion leaves out portfolio strategies starting from initial wealth \(x = 0\). Indeed, in this case, the proportion \(\pi_0\) at initial date \(t = 0\), is not defined from the amount \(\theta_0\) by \(\pi_0 = \theta_0/x\). Therefore, we have considered initial wealth \(x \in \mathbb{R}^d\).

Remark 2.2 Given an initial wealth \(x \in \mathbb{R}^d\) and a trading strategy \(\pi\), if we define

\[
\theta_t = \pi_t X_t^{x, \pi} \quad \text{for } t = 0, \ldots, T,
\]

then \(\theta^i_t, i = 1, \ldots, d\) represents the amount held in the \(i\)-th risky asset and \(\eta_t = X_t^{x, \pi} - \sum_{i=1}^d \theta^i_t\) is the amount of wealth in the riskless asset. Moreover, we have the self-financing condition written in additive form:

\[
X_{t+1}^{x, \pi} = X_t^{x, \pi} + \theta^* R_{t+1}, \quad t = 0, \ldots, T - 1.
\]

Remark 2.3 Let \(x \in \mathbb{R}^d\) be some initial wealth and consider a trading strategy \(\pi\). Then, for all \(\lambda \geq 0\), we have:

\[
X_t^{\lambda x, \pi} = \lambda X_t^{x, \pi} \quad \text{for all } t = 0, \ldots, T.
\]
We now impose some constraints on the trading strategies. Let us consider two nonempty convex sets $C^+$ and $C^-$ of $\mathbb{R}^d$ containing the origin 0, where $C^+ \neq \{0\}$. For any $x \in \mathbb{R}^*$, we say that a trading strategy is admissible, and we denote $\pi \in \mathcal{A}(x)$, if for all $t = 0, \ldots, T - 1$:

$$
\begin{align*}
\pi_t \in C^+ & \text{ if } X_t^{x,\pi} > 0, \\
\pi_t \in C^- & \text{ if } X_t^{x,\pi} < 0.
\end{align*}
$$

In other words, $C^+$ (resp. $C^-$) represents constraints on proportions when the wealth is positive (resp. negative). Such constraints cover the usual examples as described in Karatzas and Kou (1996), see section 4.

**Remark 2.4** From Remark 2.3, it is clear that $\mathcal{A}(x)$ depends only on the sign of $x$, i.e.

$$
\mathcal{A}(x) = \mathcal{A}(1) \quad \text{and} \quad \mathcal{A}(-x) = \mathcal{A}(-1) \quad \text{for all } x > 0.
$$

In the sequel, we shall denote by $\mathcal{A} = \{(x, \pi) : x \in \mathbb{R}^*, \pi \in \mathcal{A}(x)\}$.

**Definition 2.1** We say that there is no arbitrage opportunity if, for all trading strategies $(x, \pi) \in \mathcal{A}$ we have,

$$
X_T^{x,\pi} - x \geq 0 \quad P - a.s \quad \implies \quad X_T^{x,\pi} - x = 0 \quad P - a.s.
$$

Loosely speaking, an arbitrage opportunity is a way to produce nonnegative excess wealth with positive expected value out of nothing.

**Remark 2.5** Suppose that $(x, \pi) \in \mathcal{A}$ defines an arbitrage opportunity. Then for all $\lambda > 0$, it follows from Remark 2.3 that $((x/\lambda), \pi)$ also defines an arbitrage opportunity. In other words, if the financial market is not arbitrage-free then arbitrage opportunities can be realized starting from any small amount of initial wealth.

### 3 The main results

In this section, we provide a characterization of the no arbitrage condition. We first introduce some notation. We define the convex cones of $\mathbb{R}^d$:

$$
\hat{C}^+ = \left\{ x \in \mathbb{R}^d : \pi^*x \leq 0, \forall \pi \in C^+ \right\},
\hat{C}^- = \left\{ x \in \mathbb{R}^d : \pi^*x \leq 0, \forall \pi \in C^- \right\}.
$$
We also denote cone\((C^+) = \{ \lambda \pi : \lambda > 0, \pi \in C^+ \}\) and cone\((C^-) = \{ \lambda \pi : \lambda > 0, \pi \in C^- \}\).

We shall need the following assumption.

**Assumption 3.1** The sets cone\((C^+)\) and cone\((C^-)\) are closed in \(\mathbb{R}^d\).

**Remark 3.1** The assumption that cone\((C^+)\) and cone\((C^-)\) are closed is quite strong because both \(C^+\) and \(C^-\) contain the origin. Here is an example of compact convex set \(K\) in \(\mathbb{R}^2\) which contains the origin and such that cone\((K)\) is not closed: let \(K = \{(x, y) : x^2 + (y - 1)^2 \leq 1\}\), then cone\((K) = \mathbb{R} \times (0, \infty) \cup \{(0, 0)\}\). However, Assumption 3.1 is satisfied in all practical examples; see our examples section.

We also need to introduce the sets \(\mathcal{P}^+\) and \(\mathcal{P}^-\):

\[
\mathcal{P}^+ = \left\{ Q \sim P : \frac{dQ}{dP} \in L^\infty, \ R_t \in L^1(Q) \text{ and } E^Q[R_t|\mathcal{F}_{t-1}] \in \hat{C}^+, \ 1 \leq t \leq T \ P - \text{a.s.} \right\}
\]

\[
\mathcal{P}^- = \left\{ Q \sim P : \frac{dQ}{dP} \in L^\infty, \ R_t \in L^1(Q) \text{ and } E^Q[R_t|\mathcal{F}_{t-1}] \in -\hat{C}^-, \ 1 \leq t \leq T \ P - \text{a.s.} \right\}
\]

In order to establish the main result, we need the following non-degeneracy assumption.

**Assumption 3.2** Let \(t = 0, \ldots, T - 1\). Then for all \(\mathcal{F}_t\)-measurable random variables \(\varphi\) valued in \(C^+ \cup (-C^-)\),

\[
\varphi^* R_{t+1}(\omega) = 0 \implies \varphi(\omega) = 0 \quad \text{for a.e. } \omega \in \Omega.
\]

**Example 3.1** The above condition is trivially satisfied in the one-dimensional Cox-Ross-Rubinstein model with parameters \(d < 1 < u\). Indeed in this case, \(R_{t+1}\) is valued in \(\{u - 1, d - 1\}\), and \(P[R_{t+1} = u - 1|\mathcal{F}_t] > 0\) and \(P[R_{t+1} = d - 1|\mathcal{F}_t] > 0\).

**Example 3.2** Let us check Assumption 3.2 in the case of the Black and Scholes model. Suppose that \(\{S_t, t = 0, \ldots, T\}\) is a discrete sample extracted from the geometric Brownian motion \(dS_t = S_t(\mu dt + \sigma dW_t)\), where \(\mu\) and \(\sigma > 0\) are some constants. Then a straightforward calculation shows that:

\[
V[S_{t+1}|\mathcal{F}_t] = S_t^2 e^{2\mu} \left(e^{\sigma^2} - 1\right) > 0,
\]

where \(V(\cdot|\mathcal{F}_t)\) is the conditional variance operator. Now take some real-valued \(\mathcal{F}_t\)-measurable random variable \(\varphi\) such that \(\varphi R_{t+1} = 0\). Then taking conditional variance we see that \(\varphi^2 V(S_{t+1}|\mathcal{F}_t) = 0\) and thus \(\varphi = 0\).
Example 3.3 Suppose that $V[(R_{t+1}/\alpha_{t+1})|\mathcal{F}_t]$ exists and is invertible $P$-a.s. for some $\mathcal{F}_{t+1}$-measurable positive random variable $\alpha_{t+1}$. Then, considering $\varphi$ as in Assumption 3.2, we see that $V[(\varphi^* R_{t+1}/\alpha_{t+1})|\mathcal{F}_t] = \varphi^* V[R_{t+1}/\alpha_{t+1}]|\mathcal{F}_t] \varphi = 0$. Therefore $\varphi = 0$ $P$-a.s. and Assumption 3.2 holds.

We are now in a position to state the main result of this section.

Theorem 3.1 Under Assumptions 3.1 and 3.2, the following assertions are equivalent:

(i) there is no arbitrage opportunity

(ii) $\mathcal{P}^+ \neq \emptyset$ and $\mathcal{P}^- \neq \emptyset$.

The proof of this result is reported in subsection 6.1. An important (and somewhat surprising) feature of Theorem 3.1 is that the arbitrage characterization (ii) involves the constraints sets $C^+$ and $C^-$ only through the associated cones $\hat{C}^+$ and $\hat{C}^-$. Notice that $\text{cone}(C^+) = \hat{C}^+$ and $\text{cone}(C^-) = \hat{C}^-$. Therefore, the no-arbitrage characterization with constraints sets $(C^+, C^-)$ is identical to the no-arbitrage characterization with constraints sets $(\text{cone}(C^+), \text{cone}(C^-))$. This provides the following connection with the no-arbitrage condition under cone constraints; see Pham and Touzi (1999).

Corollary 3.1 Let Assumptions 3.1 and 3.2 hold. Then there is no arbitrage opportunity if and only if for all $\mathbb{H}$-adapted processes $\theta$ valued in $\text{cone}(C^+)$ or in $\text{cone}(-C^-)$ we have

$$\sum_{t=0}^{T-1} \theta^*_t R_{t+1} \geq 0 \quad P - a.s. \quad \Rightarrow \quad \sum_{t=0}^{T-1} \theta^*_t R_{t+1} = 0 \quad P - a.s.$$

Proof. Since $\text{cone}(C^+) = \hat{C}^+$ and $\text{cone}(C^-) = \hat{C}^-$, the result follows directly from the dual characterization of no-arbitrage under cone constraints in Pham and Touzi (1999). \hfill \Box

Remark 3.2 Corollary 3.1 says that our notion of no-arbitrage is in fact equivalent to the no-arbitrage condition in a financial market where the amounts of wealth invested in the risky assets are constrained to lie in $\text{cone}(C^+) \cup \text{cone}(-C^-)$. In the case $C^+ = -C^-$, this is the financial market considered in Pham and Touzi (1999).
We now provide conditions on the portfolio constraints in order to ensure that $P^+ \cap P^- \neq \emptyset$. This would guarantee the existence of some $Q \in P^+ \cap P^-$ under which any wealth process is a supermartingale; see also Remark 6.1. Furthermore, it would strengthen characterization (ii) of the no arbitrage condition in Theorem 3.1. We denote for all $t = 0, \ldots, T - 1$:

$$A_t = \{(x, \nu) \in \mathbb{R} \times L^{0,d}_t : \nu \in C^+ \text{ (resp. } C^-) \text{ if } x > 0 \text{ (resp. } < 0)\}.$$

**Assumption 3.3** For all $t = 0, \ldots, T - 1$, for all $(x_1, \nu_1) \in A_t$, $(x_2, \nu_2) \in A_t$, there exists $(x, \nu) \in A_t$ such that:

$$x \nu^* R_t \geq x_1 \nu_1^* R_t + x_2 \nu_2^* R_t.$$

**Theorem 3.2** Under Assumptions 3.1, 3.2 and 3.3, the following assertions are equivalent:

(i) there is no arbitrage opportunity

(ii) $P^+ \cap P^- \neq \emptyset$.

The proof of the above result is reported in subsection 6.2. Before concluding this section, we give sufficient conditions on the sets $C^+$ and $C^-$, inspired from Karatzas and Kou (1996), in order for Assumption 3.3 to hold.

**Proposition 3.1** Assume that the convex sets $C^+$ and $C^-$ satisfy:

$$\forall \nu^+ \in C^+, \forall \nu^- \in C^-, \lambda \nu^+ + (1 - \lambda) \nu^- \in \begin{cases} C^+ & \text{if } \lambda \geq 1, \\ C^- & \text{if } \lambda \leq 0, \end{cases} \quad (3.1)$$

$$C^+ \cup -C^- \text{ is convex.} \quad (3.2)$$

Then Assumption 3.3 holds.

**Proof.** Let $(x_1, \nu_1) \in A_t$ and $(x_2, \nu_2) \in A_t$ for some $t \in \{0, \ldots, T - 1\}$. We show that we can find $(x, \nu) \in A_t$ such that

$$x_1 \nu_1^* R_t + x_2 \nu_2^* R_t = x \nu^* R_t. \quad (3.3)$$

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(1) We first consider the case \( x_1 + x_2 \neq 0 \). We have then:

\[
x_1 \nu_1^* R_t + x_2 \nu_2^* R_t = x \nu^* R_t
\]

where \( x = x_1 + x_2 \), \( \nu = \lambda \nu_1 + (1 - \lambda) \nu_2 \) and \( \lambda = x_1/x \). If \( x_1 = 0 \) (resp. \( x_2 = 0 \)) then \( x = x_2 \) (resp. \( x = x_1 \)) and \( \nu = \nu_2 \) (resp. \( \nu = \nu_1 \)) so that \( (x, \nu) \in A_t \). It remains to see what happens when \( x_1 x_2 \neq 0 \). If \( x_1 x_2 > 0 \) then \( \lambda \in (0, 1) \). Either \( x > 0 \) and so \( \nu_1 \) and \( \nu_2 \) are valued in \( C^+ \) or \( x < 0 \) and so \( \nu_1 \) and \( \nu_2 \) are valued in \( C^- \). By the convexity of \( C^+ \) and \( C^- \), we deduce that \( (x, \nu) \in A_t \). Now assume that \( x_1 > 0 \) and \( x_2 < 0 \). We have then \( \nu_1 \in C^+ \) and \( \nu_2 \in C^- \). Either \( x > 0 \) and so \( \lambda > 1 \) or \( x < 0 \) and so \( \lambda < 0 \). It follows from (3.1) that \( (x, \nu) \in A_t \). By analogy, we show that \( (x, \nu) \in A_t \) if \( x_1 < 0 \) and \( x_2 > 0 \).

(2) Consider finally the case \( x_1 + x_2 = 0 \). Assume first \( x_1 > 0 \). Then \( \nu_1 \in C^+ \), \( \nu_2 \in C^- \) and by \( \frac{1}{2} \nu_1 - \frac{1}{2} \nu_2 \in C^+ \cup -C^- \) by (3.2). If \( \frac{1}{2} \nu_1 - \frac{1}{2} \nu_2 \in C^+ \), then we can write

\[
x_1 \nu_1^* R_t + x_2 \nu_2^* R_t = x \nu^* R_t
\]

with \( x = 2x_1 > 0 \), \( \nu = \frac{1}{2} \nu_1 - \frac{1}{2} \nu_2 \in C^+ \) and so \( (x, \nu) \in A_t \). Similarly, if \( \frac{1}{2} \nu_1 - \frac{1}{2} \nu_2 \in -C^- \), we can write

\[
x_1 \nu_1^* R_t + x_2 \nu_2^* R_t = x \nu^* R_t
\]

with \( x = 2x_2 < 0 \), \( \nu = \frac{1}{2} \nu_2 - \frac{1}{2} \nu_1 \in C^- \) and so \( (x, \nu) \in A_t \). By analogy, we show that if \( x_1 < 0 \) then we have a decomposition of the form (3.3) with \( (x, \nu) \in A_t \). Finally assume that \( x_1 = 0 \) and so \( x_2 = 0 \). Therefore (3.3) is satisfied with \( (0, \nu) \in A_t \) for any \( \nu \in L_t^{0,d} \).  

4 Examples

In all following examples, Assumption 3.1 is satisfied.

1. **Unconstrained case**: this corresponds to the classical incomplete market framework \( C^+ = C^- = R^d \). Then, \( \text{cone}(C^+) = \text{cone}(C^-) = R^d \), and \( \hat{C}^+ = \hat{C}^- = \{0\} \). Theorem 3.1 reduces to the classical characterization of the no-arbitrage condition by the existence of an equivalent martingale measure for the discounted price process \( S \).
2. **Prohibition of short-selling of stocks**: this corresponds to the case $C^+ = -C^- = [0, \infty)^d$. Then $\text{cone}(C^+) = \text{cone}(-C^-) = [0, +\infty)^d$ and $\hat{C}^+ = -\hat{C}^- = (-\infty, 0]^d$. Theorem 3.1 reduces to the characterization of the no-arbitrage condition by the existence of an equivalent probability measure under which the price process $S$ is a supermartingale; see Jouni and Kallal (1995).

3. **Constraints on the short-selling of stocks**: $C^+ = \prod_{i=1}^d [-k_i, \infty)$, $C^- = \prod_{i=1}^d (-\infty, l_i]$ with $k_i, l_i > 0$ for $i = 1, \ldots, d$. Then $\text{cone}(C^+) = \text{cone}(C^-) = \mathbb{R}^d$ and $\hat{C}^+ = \hat{C}^- = \{0\}$. In this context, Theorem 3.1 says that the no-arbitrage condition is equivalent to the existence of an equivalent martingale measure for the price process $S$. Surprisingly, we obtain the same characterization as in the unconstrained case.

4. **Rectangular constraints**: $C^+ = \prod_{i=1}^d [-k_i, l_i]$ with $k_i, l_i > 0$ for $i = 1, \ldots, d$. Then $\text{cone}(C^+) = \text{cone}(C^-) = \mathbb{R}^d$ and $\hat{C}^+ = \hat{C}^- = \{0\}$. As in the previous example, we obtain the same characterization of no-arbitrage as in the unconstrained case.

5. **Incomplete and constrained market with prohibition of short-selling of the first $n_1$ stocks, prohibition of being long in the next $n_2$ stocks and prohibition of investment in the next $n_3$ stocks ($n_1 + n_2 + n_3 \leq d$)**: $C^+ = -C^- = C = \{\pi \in \mathbb{R}^d : \pi_i \geq 0, i = 1, \ldots, n_1, \pi_i \leq 0, i = n_1 + 1, \ldots, n_1 + n_2, \pi_i = 0, i = n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3\}$. Then $\text{cone}(C) = C$ and $\hat{C} = \{x \in \mathbb{R}^d : x_i \leq 0, i = 1, \ldots, n_1, x_i \geq 0, i = n_1 + 1, \ldots, n_1 + n_2, x_i = 0, i = n_1 + n_2 + n_3, \ldots, d\}$. From Theorem 3.1, the no-arbitrage condition is characterized by the existence of an equivalent probability measure under which:

- the first $n_1$ components of the price process (assets subject to no short-selling constraint) are supermartingale,
- the next $n_2$ components of the price process are submartingale,
- the last $d - n_1 - n_2 - n_3$ components of the price process (unconstrained assets) are martingale.

6. **Both $C^+$ and $C^-$ are closed convex cones with vertex at zero**: this clearly generalizes all the previous cases except 3 and 4. Then Theorem 3.1 provides an extension
to the result of Pham and Touzi (1999).

7. **Constraints on borrowing**: in this example, we put constraints on the proportion of wealth invested in the non-risky asset $1 - \sum_{i=1}^{d} \pi_i$. Let $C^+ = \{ \pi \in \mathbb{R}^d : \sum_{i=1}^{d} \pi_i \leq k \}$ and $C^- = \{ \pi \in \mathbb{R}^d : \sum_{i=1}^{d} \pi_i \geq l \}$ for some $k \geq 1$ and $l \leq 0$. Then cone$(C^+) = \text{cone}(C^-) = \mathbb{R}^d$ and $\hat{C}^+ = \hat{C}^- = \{0\}$. We find again that the no-arbitrage characterization is the same as in the unconstrained case.

**Remark 4.1** Conditions (3.1) and (3.2) are satisfied in the context of the above examples for the cases 1, 2, 5 and 6 with $C^+ = -C^-$. Then Assumption 3.3 holds.

## 5 Connection with the problem of super-replication

In this section, we study the special case

$$C := C^+ = -C^-.$$  

We also denote $\mathcal{P} := \mathcal{P}^+ = \mathcal{P}^-$. We intend to relate our result to the dual formulation of the super-replication cost of contingent claims derived by Föllmer and Kramkov (1997). As in Föllmer and Kramkov (1997), we consider a non-negativity constraint on the wealth process. We therefore define the set of strategies:

$$\mathcal{A}^+ := \{ \pi \in \mathcal{A}(1) : X^{1,\pi}(.) \geq 0 \ P - \text{a.s.}\} = \{ \pi \text{ IF - adapted valued in } C^+ : X^{1,\pi}(.) \geq 0 \ P - \text{a.s.}\}$$

Let $B$ be a contingent claim, i.e., a nonnegative $\mathcal{F}_T$-measurable random variable. The super-replication cost of $B$ is defined by

$$p(B) = \inf\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A}^+, \ X_T^{x,\pi} \geq B \ P - \text{a.s}\},$$

i.e., the minimal initial capital needed for hedging without risk the contingent claim $B$. We first recall the result of Föllmer and Kramkov (1997). For all $\pi \in \mathcal{A}^+$, we denote by $Y^{\pi}$ the process

$$Y^{\pi}_t = \sum_{u=0}^{t-1} \pi^*_u R_{u+1}, \quad t = 0, \ldots, T,$$
so that the wealth process $X^{x,\pi} = x\mathcal{E}(Y^{\pi})$, where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential.

Next, we introduce the sets

$$Y := \{Y^{\pi} : \pi \in \mathcal{A}^+\}$$

and

$$\mathcal{P}(Y) := \{Q \sim P : \exists A \in \mathcal{I}, \ Y - A \text{ is } Q \text{– supermartingale for all } Y \in \mathcal{Y}\},$$

where $\mathcal{I}$ is the set of all adapted nondecreasing processes. For all measures $Q \in \mathcal{P}(Y)$ we denote by $A^Y(Q)$ the upper variation process of $\mathcal{Y}$ under $Q$ as defined in Föllmer and Kramkov (1997). Then we have

**Theorem 5.1** (Föllmer and Kramkov 1997). Assume that $\mathcal{P}(Y) \neq \emptyset$. Then

$$p(B) = \sup_{Q \in \mathcal{P}(Y)} E^Q \left[ B\mathcal{E}(A^Y(Q))^{-1} \right].$$

In the previous literature on the super-replication problem without constraints, the dual formulation is obtained under the no-arbitrage condition. Our purpose is to relate the condition $\mathcal{P}(Y) \neq \emptyset$ to the no-arbitrage concept.

**Proposition 5.1** (i) $\mathcal{P} \subset \mathcal{P}(Y)$.

(ii) Suppose that there exists no arbitrage opportunity. Then $\mathcal{P}(Y) \neq \emptyset$ and $p(1) = 1$.

(iii) Suppose that $\mathcal{P}(Y) \neq \emptyset$ and $p(1) = 1$. Then there exists no arbitrage opportunity with nonnegative wealth, i.e.

$$X^{1,\pi} \geq 1 \text{ P – a.s. for some } \pi \in \mathcal{A}^+ \implies X_T^{1,\pi} = 1 \text{ P – a.s.}$$

**Proof.** (i) is trivial since for all $Q \in \mathcal{P}$ and $\pi \in \mathcal{A}^+$, the process $Y^{\pi}$ is a supermartingale under $Q$.

We now prove (ii). Suppose that there is no arbitrage opportunity. Then $\mathcal{P} \neq \emptyset$ by Theorem 3.1 and therefore $\mathcal{P}(Y) \neq \emptyset$ by (i). Next, since $0 \in \mathcal{A}^+$, we see that $p(1) \leq 1$. On the other hand, let $Q$ be any probability measure in $\mathcal{P}$, then it is easily checked that $X^{x,\pi}$ is a $Q$-supermartingale for all $(x, \pi) \in (0, \infty) \times \mathcal{A}^+$, and therefore $p(1) \geq 1$. 

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To see that (iii) holds, we argue by contradiction. Suppose that

\[ X^{1,\pi}_T \geq 1 \quad P \text{-- a.s.} \]

for some \( \pi \in \mathcal{A}^+ \). Set \( B := X^{1,\pi}_T - 1 \) and assume that \( P[B > 0] > 0 \). By definition of the super-replication of the contingent claim \( 1 + B \) and from Theorem 5.1, we have:

\[
1 \geq p(1 + B) \\
= \sup_{Q \in \mathcal{P}(\mathcal{Y})} E^Q [(1 + B)E(A^Y(Q))^{-1}] \\
= 1 + \sup_{Q \in \mathcal{P}(\mathcal{Y})} E^Q [B E(A^Y(Q))^{-1}] \\
> 1,
\]

where we used the fact that \( p(1) = 1, B \geq 0 \) and \( P[B > 0] > 0 \). This provides the required contradiction. \( \square \)

6  Proof of the main results

6.1  Proof of Theorem 3.1

To prove Theorem 3.1, we need some preliminary results.

**Lemma 6.1** Assume that \( \mathcal{P}^+ \) and \( \mathcal{P}^- \) are not empty. Then for all \((x, \pi) \in \mathcal{A}\), there exists a probability measure \( Q^{x,\pi} \) equivalent to \( P \) such that the wealth process \( \{X^{x,\pi}_t, t = 0, \ldots, T\} \) is a supermartingale under \( Q^{x,\pi} \).

**Proof.** Consider some \( Q^+ \) (resp. \( Q^- \)) in \( \mathcal{P}^+ \) (resp. \( \mathcal{P}^- \)) and denote by \( Z^+ \) (resp. \( Z^- \)) their Radon-Nikodym density with respect to \( P \). Let \((x, \pi) \in \mathcal{A}\) and \( X^{x,\pi} \) the associated wealth process defined by \( X^{x,\pi}_0 = x \) and for all \( t = 0, \ldots, T - 1 \):

\[
X^{x,\pi}_{t+1} = X^{x,\pi}_t + X^{x,\pi}_t \pi^*_t R_{t+1}.
\]

Define then the positive \( \mathcal{F}_T \)-measurable random variable \( Z^{x,\pi} \) by:

\[
Z^{x,\pi} = \prod_{t=0}^{T-1} Z^{x,\pi}_{t,t+1}
\]
where
\[ Z_{t,t+1}^{x,\pi} = \frac{E[Z^+|\mathcal{F}_{t+1}]}{E[Z^+|\mathcal{F}_t]} 1_{\{X_{t+1}^{x,\pi} \geq 0\}} + \frac{E[Z^-|\mathcal{F}_{t+1}]}{E[Z^-|\mathcal{F}_t]} 1_{\{X_{t+1}^{x,\pi} < 0\}}. \]

Notice that
\[ E[Z_{t,t+1}^{x,\pi}|\mathcal{F}_t] = 1. \tag{6.2} \]

Let \( Q_{t,t+1}^{x,\pi} \) be the probability measure equivalent to \( P \) with density \( Z_{t,t+1}^{x,\pi} \). We first check that \( E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] \) exists. To see this, denote by \( Z_{t,t+1}^{x,\pi} := \prod_{u=0}^{t-1} Z_{u,u+1}^{x,\pi} \). Then, by Bayes rule,
\[ E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] = \frac{E[Z_{t,t+1}^{x,\pi}(Z_{t,t+1}^{x,\pi}/Z_{t,t+1}^{x,\pi})|R_{t+1}|\mathcal{F}_t]}{E[Z_{t,t+1}^{x,\pi}(Z_{t,t+1}^{x,\pi}/Z_{t,t+1}^{x,\pi})|\mathcal{F}_t]} \]
\[ = \frac{E[(Z_{t,t+1}^{x,\pi}/Z_{t,t+1}^{x,\pi})|R_{t+1}|\mathcal{F}_t]}{E[(Z_{t,t+1}^{x,\pi}/Z_{t,t+1}^{x,\pi})|\mathcal{F}_t]} \]

since all random variables inside the expectation are nonnegative. By the definition of \( Z_{t,t+1}^{x,\pi} \) and using Bayes rule and the law of iterated expectations, we see that:
\[ E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] = E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] = \frac{1}{2} \{X_{t+1}^{x,\pi} \geq 0\} E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] + \frac{1}{2} \{X_{t+1}^{x,\pi} < 0\} E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] < \infty \]

since \( R_t \in L^1(Q^+) \cap L^1(Q^-) \). Thus, we get:
\[ X_{t+1}^{x,\pi} \pi_t E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] = X_{t+1}^{x,\pi} \pi_t E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] + X_{t+1}^{x,\pi} \pi_t E_{Q_{t,t+1}^{x,\pi}}[R_{t+1}|\mathcal{F}_t] < 0 \]

by definition of the sets \( \mathcal{P}^+ \) and \( \mathcal{P}^- \). Plugging the last inequality in (6.1), we obtain that
\[ E_{Q_{t,t+1}^{x,\pi}}[X_{t+1}^{x,\pi}|\mathcal{F}_t] \leq X_t^{x,\pi}, \]
for all \( t = 0, \ldots, T-1 \), which proves the supermartingale property of \( X^{x,\pi} \) under \( Q_{t,t+1}^{x,\pi} \). \( \square \)

**Remark 6.1** Assume that \( \mathcal{P}^+ \cap \mathcal{P}^- \neq \emptyset \) and let \( Q \in \mathcal{P}^+ \cap \mathcal{P}^- \) and \( Z \) its Radon-Nikodym density with respect to \( P \). Then in the proof of Lemma 6.1, we can choose \( Z_{t,t+1}^{x,\pi} = Z \) for all \((x, \pi) \in \mathcal{A}\). It follows that for all \((x, \pi) \in \mathcal{A}\), the wealth process \( X^{x,\pi} \) is a \( Q \)-supermartingale for any \( Q \in \mathcal{P}^+ \cap \mathcal{P}^- \).
Let us now define for all \( t = 0, \ldots, T - 1 \):

\[
\mathcal{A}_t^+ = \\{ \nu \in L_t^{0,d} : \nu \in C^+ \},
\]

\[
\mathcal{A}_t^- = \\{ \nu \in L_t^{0,d} : \nu \in C^- \}.
\]

As in Kabanov and Kramkov (1994), we consider the bounded \( \mathbb{R}^d \)-valued random variable

\[
\delta_t = R_t / (1 + |R_t|), \quad \text{for all } t = 0, \ldots, T.
\]

**Remark 6.2** Notice that \( V(\delta_{t+1}|\mathcal{F}_t) \) exists and, therefore, a sufficient condition for Assumption 3.2 to be satisfied is that \( V(\delta_{t+1}|\mathcal{F}_t) \) is invertible \( P \)-a.s. for all \( t = 0, \ldots, T - 1 \); see Example 3.3.

Next, for all \( t = 1, \ldots, T \), define the sets:

\[
\mathcal{B}_t^+ = \\{ U \in L_t^0 : \exists x > 0 \text{ and } \nu \in \mathcal{A}_{t-1}^+, x\nu^*\delta_t \geq U \},
\]

\[
\mathcal{B}_t^- = \\{ U \in L_t^0 : \exists x < 0 \text{ and } \nu \in \mathcal{A}_{t-1}^-, x\nu^*\delta_t \geq U \}.
\]

Notice that \( \mathcal{B}_t^+ \) and \( \mathcal{B}_t^- \) contain the origin since \( 0 \in C^+ \cap C^- \). It is easily checked that \( \mathcal{B}_t^+ \) and \( \mathcal{B}_t^- \) are two convex cones of \( L_t^0 \). The cone property is immediate. To check the convexity property, consider some \( U_1, U_2 \in \mathcal{B}_t^+ \) (resp. \( \mathcal{B}_t^- \)), then there exist some positive (resp. negative) real \( x_1, x_2 \) and \( \nu_1, \nu_2 \in \mathcal{A}_{t-1}^+ \) (resp. \( \mathcal{A}_{t-1}^- \)) such that \( x\nu^*\delta_t \geq U_1 + U_2 \) where \( x = x_1 + x_2, \nu = \lambda \nu_1 + (1 - \lambda)\nu_2 \) and \( \lambda = x_1/x \). The convexity of \( C^+ \) (resp. \( C^- \)) implies that \( U_1 + U_2 \in \mathcal{B}_t^+ \) (resp. \( \mathcal{B}_t^- \)).

**Lemma 6.2** Suppose that there is no arbitrage opportunity. Then for all \( t = 1, \ldots, T \), we have \( \mathcal{B}_t^+ \cap L_t^{0+} = \{0\} \) and \( \mathcal{B}_t^- \cap L_t^{0+} = \{0\} \).

**Proof.** We argue by contradiction. Assume that \( \mathcal{B}_t^+ \cap L_t^{0+} \neq \{0\} \) (resp. \( \mathcal{B}_t^- \cap L_t^{0+} \neq \{0\} \)) for some \( t \in \{1, \ldots, T\} \). Then there exist \( U \in L_t^{0+}, U \neq 0, x > 0 \) (resp. \( x < 0 \)) and \( \nu \in \mathcal{A}_{t-1}^+ \) (resp. \( \nu \in \mathcal{A}_{t-1}^- \)) such that \( x\nu^*\delta_t \geq U \) \( P \)-a.s. and then

\[
x\nu^*R_t \geq 0, \quad P - \text{a.s.} \quad \text{and} \quad P[x\nu^*R_t > 0] > 0.
\]

(6.3)
Consider the trading strategy \( \pi \) defined by
\[
\pi_u = 0 \quad \text{for} \quad u \neq t - 1 \quad \text{and} \quad \pi_{t-1} = \nu.
\]
The associated wealth process is:
\[
X^{x,\pi}_u = x, \quad u = 0, \ldots, t - 1,
\]
\[
X^{x,\pi}_u = x + xu^*R_t, \quad u = t, \ldots, T.
\]
It follows that \((x, \pi) \in A \) (recall that \( 0 \in C^+ \cap C^- \)). Moreover by (6.3), we have:
\[
X_T^{x,\pi} \geq x, \quad P - \text{a.s.} \quad \text{and} \quad P[X_T^{x,\pi} > x] > 0,
\]
which is an arbitrage opportunity.

\[ \square \]

**Lemma 6.3** Under Assumptions 3.1 and 3.2, for all \( t = 1, \ldots, T \), the following implications hold
\[
\mathcal{B}_t^+ \cap L_{t+1}^1 = \{0\} \implies \overline{\mathcal{B}_t^+ \cap L_{t+1}^1} = \{0\},
\]
\[
\mathcal{B}_t^- \cap L_{t+1}^1 = \{0\} \implies \overline{\mathcal{B}_t^- \cap L_{t+1}^1} = \{0\},
\]
where \( \overline{\text{cl}} \) is the closure in the sense of the \( L^1 \) topology.

**Proof.** We only prove the first implication since the second one is proved by the same way.

Let \( U \in \overline{\text{cl}}(\mathcal{B}_t^+ \cap L_{t+1}^1) \cap L_{t+1}^1. \) Then there exists a sequence \((U^n)_n \subset \mathcal{B}_t^+ \cap L_{t+1}^1\) converging to \( U \) in the sense of the \( L^1 \) topology, and therefore \( P\)-a.s, possibly along a subsequence. By definition of \( \mathcal{B}_t^+ \), for each \( n \in \mathbb{N} \), there exists \( x^n > 0, \nu^n \in \mathcal{A}_{t-1}^+ \) such that:
\[
x^n\nu^n\delta_t \geq U^n. \quad (6.4)
\]

In order to prove the required result, we intend to show that \((|x^n\nu^n|)_n\) converges to zero in probability. Then along a subsequence it also converges almost surely to zero. Taking almost sure limit in inequality (6.4), we find that \( U \) is nonpositive. Recalling that \( U \) is by assumption nonnegative, we conclude that \( U = 0 \).
We adapt the arguments of Kabanov and Kramkov (1994). Fix some \( \varepsilon > 0 \) and let \( \varphi^n = \frac{x^n \nu^n}{|x^n \nu^n|} 1_{|x^n \nu^n| \geq \varepsilon} \). Let \( \mathcal{E} = \{-1,+1\}^d \) and for all \( e \in \mathcal{E} \), \( A_e = \{ z \in \mathbb{R}^d : z^i \geq 0 \text{ iff } e^i = +1 \} \).

We consider the sequence defined by \( \varphi^n_e = \varphi^n 1_{\nu^n \in A_e} \). It follows that for all \( e \in \mathcal{E} \), \( A_e = \{ z \in \mathbb{R}^d : z^i \geq 0 \text{ iff } e^i = +1 \} \),

\[
P[|x^n \nu^n| \geq \varepsilon] = E[1_{|x^n \nu^n| \geq \varepsilon}] = E|\varphi^n| = E \left[ |\varphi^n| \sum_{e \in \mathcal{E}} 1_{\nu^n \in A_e} \right] = E \left[ \sum_{e \in \mathcal{E}} |\varphi^n_e| \right] = \sum_{e \in \mathcal{E}} E|\varphi^n_e| \quad (6.5)
\]

For each \( e \in \mathcal{E} \), the sequence \( (\varphi^n_e)_n \) is bounded by 1 uniformly in \( \omega \in \Omega \). Since \( L^2_{t-1} \) is the dual space of \( L^2_{t-1} \), which is a Banach separable space, \( (\varphi^n_e)_n \) converges (possibly along a subsequence) in the sense of the weak topology \( \sigma(L^2_{t-1}, L^2_{t-1}) \) to some \( \varphi_e \in L^2_{t-1} \). Next, we prove that \( (\varphi^n_e)_n \) converges to \( \varphi_e \) in the sense of the weak* topology \( \sigma(L^\infty_{t-1}, L^1_{t-1}) \).

Let \( \psi \in L^1_{t-1} \), we want to show that \( E[\psi \varphi^n_e] \) converges to \( E[\psi \varphi_e] \). Recalling that the space \( L^2_{t-1} \) is dense in the space \( L^1_{t-1} \), there exists a sequence \( (\psi^p)_p \subset L^2_{t-1} \) converging to \( \psi \) in the sense of the \( L^1_{t-1} \) norm. Let \( \varepsilon > 0 \), then there exists \( p^* \) such that for all \( p \geq p^* \), \( E[|\psi^p - \psi|] < \varepsilon/3 \).

Since \( (\varphi^n_e)_n \) converges to \( \varphi_e \) in the sense of the weak topology \( \sigma(L^2_{t-1}, L^1_{t-1}) \) and \( \psi^p \in L^2_{t-1} \), there exists \( N_{p^*} \), such that for all \( n \geq N_{p^*} \), \( E[|\psi^p \varphi^n_e|] - E[|\psi^p \varphi_e|] \) \( \leq \varepsilon/3 \). Next, recalling that \( (\varphi^n_e)_n \) is bounded by 1 uniformly in \( \omega \in \Omega \), it is straightforward to see that \( \varphi_e \) is also bounded by 1 uniformly in \( \omega \in \Omega \). Now, notice that,

\[
|E[\psi \varphi^n_e] - E[\psi \varphi_e]| \leq |E[\psi \varphi^n_e] - E[\psi^p \varphi^n_e]| + |E[\psi^p \varphi^n_e] - E[\psi^p \varphi_e]| + |E[\psi^p \varphi_e] - E[\psi \varphi_e]| \leq 2E[|\psi^p - \psi|] + |E[\psi^p \varphi^n_e] - E[\psi^p \varphi_e]|.
\]

It follows that for all \( n \geq N_{p^*} \), \( E[|\psi \varphi^n_e| - E[\psi \varphi_e]| \leq \varepsilon \). This prove that the sequence \( (\varphi^n_e)_n \) converges to \( \varphi_e \) in the sense of the weak* topology \( \sigma(L^\infty_{t-1}, L^1_{t-1}) \) and \( \varphi_e \in L^\infty_{t-1} \). Furthermore, \( \varphi_e \in A_e \cup \{0\} \), \( P \)-a.s. Next, for all \( e \in A_e \), \( E[e^* \varphi^n_e] \) converges towards \( E[e^* \varphi_e] \) as \( n \) goes to infinity. Recalling the definition of \( \mathcal{E} \), we see that \( e^* \varphi^n_e \) \( = |\varphi^n_e| \) and \( e^* \varphi_e \) \( = |\varphi_e| \).

Sending \( n \) to infinity in (6.5), we have:

\[
\liminf_{n \to \infty} P[|x^n \nu^n| \geq \varepsilon] = \sum_{e \in \mathcal{E}} E[|\varphi^n_e|]. \quad (6.6)
\]
Now, suppose that \( \liminf_{n \to \infty} P[(|x^n\nu^n| \geq \varepsilon)] > 0 \), then by (6.6) there exists \( e \in \mathcal{E} \) such that \( E|\varphi_e| \neq 0 \). By (6.4), we have:

\[
\varphi_e^* \delta_t \geq \alpha^n U^n.
\]  

(6.7)

where \( \alpha^n = \frac{1}{|x^n\nu^n|}1_{\{|x^n\nu^n|\geq \varepsilon\}}1_{\{e^n \in A_e\}} \). The sequence \( (\alpha^n)_n \) is bounded by \( 1/\varepsilon \) uniformly in \( \omega \in \Omega \). Using the same line of argument as before, it converges (possibly along a subsequence) in the sense of the weak* topology \( \sigma(L_{t-1}^\infty, L_{t-1}^1) \) to some \( \alpha \in L_{t-1}^\infty \). Notice also that for all \( n \), \( \alpha^n \) is non-negative and therefore \( \alpha \) is non-negative.

Recalling that \( U^n \) converges to \( U \) in the sense of the \( L^1 \)-convergence, we deduce by Proposition III.12 (iv) of Brézis (1983) that for any bounded \( \mathcal{F}_t \)-measurable random variable \( \xi \geq 0 \), \( E[\xi U^n\alpha^n] \) converges to \( E[\xi U\alpha] \). On the other hand, since \( \delta_t \) belongs to \( L^1 \) and \( \varphi_e^n \) converges in the sense the weak* topology \( \sigma(L_{t-1}^\infty, L_{t-1}^1) \) to some \( \varphi_e \), we have that \( E[\xi \varphi_e^* \delta_t] \) converges towards \( E[\xi \varphi_e^* \delta_t] \) as \( n \) goes to infinity for any bounded \( \mathcal{F}_t \)-measurable random variable \( \xi \geq 0 \). By sending \( n \) to infinity in (6.7), we have then:

\[
E[\xi \varphi_e^* \delta_t] \geq E[\xi U\alpha] \geq 0.
\]

From the arbitrariness of the \( \mathcal{F}_t \)-measurable random variable \( \xi \geq 0 \), we deduce that \( \varphi_e^* \delta_t \geq 0 \).

Recalling that \( \varphi_e^n = \frac{x^n\nu^n}{|x^n\nu^n|}1_{\{|x^n\nu^n|\geq \varepsilon\}}1_{\{e^n \in A_e\}} \), we have that \( \varphi_e^n \) belongs to \( \text{cone}(C^+) \). From Assumption 3.1, \( \text{cone}(C^+) \) is a closed set in \( B^d \) and it follows that \( \varphi_e \in \text{cone}(C^+) \). Then there exists a positive random variable \( \lambda_e \) and \( \nu_e \in A_{t-1}^+ \) such that \( \varphi_e = \lambda_e \nu_e \). Now, let \( y_e = \max(\lambda_e, 1) \) and \( \pi_e = \min(\lambda_e, 1) \nu_e = \min(\lambda_e, 1) \nu_e + (1-\min(\lambda_e, 1))0 \in C^+ \). Then, we have \( \varphi_e = y_e \pi_e \) with \( y_e \geq 1 \) and \( \pi_e \in L_{t-1}^\infty \). Let \( x_e = \inf \{ \lambda \geq 1 / (\varphi_e/\lambda) \in C^+ \} \). From the existence of \( y_e \), we see that \( x_e \) is well-defined. Notice that for all \( a \geq 1 \), \( \{x_e \leq a\} = \{\varphi_e/a \in C^+\} \).

The first inclusion is from \( \varphi_e/a = (\varphi_e/x_e)(x_e/a) + 0(1-x_e/a) \) and 0 belongs to the convex set \( C^+ \). The reverse inclusion is trivial from the definition of \( x_e \). Recalling that \( \varphi_e \in \mathcal{F}_{t-1} \), this proves that \( x_e \) is \( \mathcal{F}_{t-1} \) measurable.

Let \( \pi_e = (\varphi_e/x_e) \), then \( \pi_e^* \delta_t \in B_t^+ \cap L_t^{1+} = \{0\} \) and therefore \( \pi_e^* R_t = 0 \). By Assumption 3.1, this implies that \( \pi_e = 0 \) and therefore \( \varphi_e = 0 \) which contradicts the fact that \( E|\varphi_e| \neq 0 \).

\( \square \)
Lemma 6.4 Suppose that there is no arbitrage opportunity. Then under Assumptions 3.1 and 3.2, for all $t = 1, \ldots, T$, there exist $Z^+_t$ and $Z^-_t \in L^\infty_t$ with $Z^+_t > 0$, $Z^-_t > 0$ $P$-a.s such that:

$$E\left[Z^+_t R_t|\mathcal{F}_{t-1}\right] \in \hat{C}^+ \quad \text{and} \quad E\left[Z^-_t R_t|\mathcal{F}_{t-1}\right] \in -\hat{C}^-, \quad P - a.s.$$ 

Proof. From Lemmas 6.2 and 6.3, we have $\text{cl}(B^+_t \cap L^1_t) \cap L^1_t = \{0\}$ and $\text{cl}(B^-_t \cap L^1_t) \cap L^1_t = \{0\}$. Moreover, $B^+_t$ and $B^-_t$ contain $L^1_t$ because $C^+$ and $C^-$ contain zero. From Yan’s Theorem (see Yan 1980), we deduce that there exist $\hat{Z}^+_t$ and $\hat{Z}^-_t \in L^\infty_t$ with $\hat{Z}^+_t > 0$, $\hat{Z}^-_t > 0$ $P$-a.s. such that:

$$E\left[\hat{Z}^+_t U\right] \leq 0, \quad \text{for all} \quad U \in B^+_t \cap L^1_t, \quad (6.8)$$

$$E\left[\hat{Z}^-_t U\right] \leq 0, \quad \text{for all} \quad U \in B^-_t \cap L^1_t. \quad (6.9)$$

Let $z \in C^+$ (resp. $C^-$), $t \in \{1, \ldots, T\}$, $B \in \mathcal{F}_{t-1}$ and $\nu = z1_B$. Then $\nu \in A^+_t$ (resp. $A^-_t$) and $U = \nu^* \delta_t$ (resp. $-\nu^* \delta_t$) $\in B^+_t \cap L^1_t$ (resp. $B^-_t \cap L^1_t$). The separation inequalities (6.8)-(6.9) and the arbitrariness of $B \in \mathcal{F}_{t-1}$ imply therefore:

$$z^* E\left[Z^+_t R_t|\mathcal{F}_{t-1}\right] \leq 0, \quad \forall z \in C^+, \quad (6.8)$$

$$-z^* E\left[Z^-_t R_t|\mathcal{F}_{t-1}\right] \leq 0, \quad \forall z \in C^- \quad (6.9)$$

Let $Z^+_t = \hat{Z}^+_t/(1 + |R_t|)$ and $Z^-_t = \hat{Z}^-_t/(1 + |R_t|)$ are positive $P$-a.s. and lie in $L^\infty_t$. Then $E[Z^+_t R_t|\mathcal{F}_{t-1}] = E[\hat{Z}^+_t \delta_t|\mathcal{F}_{t-1}]$ and $E[Z^-_t R_t|\mathcal{F}_{t-1}] = E[\hat{Z}^-_t \delta_t|\mathcal{F}_{t-1}]$ are well-defined and finited, and $E[Z^+_t R_t|\mathcal{F}_{t-1}] \in \hat{C}^+$ and $E[Z^-_t R_t|\mathcal{F}_{t-1}] \in -\hat{C}^-, \quad P$-a.s. \hfill $\Box$

Proof of Theorem 3.1

We first prove the implication $(ii) \implies (i)$. Let $(x, \pi) \in \mathcal{A}$ and $X^{x,\pi}$ its wealth process. By Lemma 6.1, there exists a probability measure $Q^{x,\pi}$ equivalent to $P$ such that

$$E^{Q^{x,\pi}}[X^{x,\pi}_T] \leq x.$$ 

Since $Q^{x,\pi}$ is equivalent to $P$, this last relation shows that $(x, \pi)$ cannot be an arbitrage opportunity.
We now prove the implication (i) $\implies$ (ii). From Lemma 6.4, there exist $Z^+_T$ and $Z^-_T \in L^\infty$ with $Z^+_T > 0$ and $Z^-_T > 0$, P.a.s such that $E[Z^+_T R_T | \mathcal{F}_{T-1}] \in \hat{C}^+$ and $E[Z^-_T R_T | \mathcal{F}_{T-1}] \in -\hat{C}^-$, P.a.s. Obviously, we can assume without loss of generality that $E[Z^+_T] = E[Z^-_T] = 1$. Then, we define the probability measure $Q^+_T$ (resp. $Q^-_T$) equivalent to $P$ with density $Z^+_T$ (resp. $Z^-_T$). Notice that the statement of Lemma 6.4 is valid for any complete probability space $(\Omega, \mathcal{F}, \mathcal{IF}, Q)$ with a probability measure $Q$ equivalent to $P$. Applying Lemma 6.4 to $(\Omega, \mathcal{F}, \mathcal{IF}, Q^+_T)$ and $(\Omega, \mathcal{F}, \mathcal{IF}, Q^-_T)$, we see that there exist $Z^+_{T-1}$ and $Z^-_{T-1} \in L^\infty$ with $Z^+_{T-1} > 0$ and $Z^-_{T-1} > 0$, P.a.s such that $E^{Q^+_T}[Z^+_{T-1} R_{T-1} | \mathcal{F}_{T-2}] \in \hat{C}^+$ and $E^{Q^-_T}[Z^-_{T-1} R_{T-1} | \mathcal{F}_{T-2}] \in -\hat{C}^-$, P.a.s. This leads to $E[Z^+_T Z^+_{T-1} R_{T-1} | \mathcal{F}_{T-2}] \in \hat{C}^+$ and $E[Z^-_T Z^-_{T-1} R_{T-1} | \mathcal{F}_{T-2}] \in -\hat{C}^-$, P.a.s. Without loss of generality, we can assume that $E[Z^+_{T-1} Z^+_T] = 1$ and $E[Z^-_{T-1} Z^-_T] = 1$.

By induction, using Lemma 6.4, we construct for all $t = 1, \ldots, T$, $Z_t^+$ and $Z_t^- \in L^\infty$ with $Z_t^+ > 0$, $Z_t^- > 0$ P.a.s such that:

$$E[Z^+_t \cdots Z^+_1] = 1 \quad \text{and} \quad E[Z^-_t \cdots Z^-_1] = 1$$

$$E\left[Z^+_t \cdots Z^+_1 R_t | \mathcal{F}_{t-1}\right] \in \hat{C}^+ \quad \text{and} \quad E\left[Z^-_t \cdots Z^-_1 R_t | \mathcal{F}_{t-1}\right] \in -\hat{C}^-, \quad \text{P.a.s. (6.10)}$$

We then define the probability measures $Q^+$ and $Q^-$ equivalent to P by their Radon-Nikodym densities:

$$\frac{dQ^+}{dP} = \prod_{t=1}^T Z^+_t \quad \text{and} \quad \frac{dQ^-}{dP} = \prod_{t=1}^T Z^-_t.$$ 

It follows that $dQ^+/dP$ and $dQ^-/dP \in L^\infty$. By Bayes rule, we have for all $t = 1, \ldots, T$,

$$E^{Q^+}[|R_t|] = E\left[Z^+_t \cdots Z^+_1 | R_t\right] = E\left[Z^-_t \cdots Z^-_1 E\left[Z^+_t \cdots Z^+_1 | R_t, | \mathcal{F}_{t-1}\right]\right] < \infty.$$ 

By a similar argument we also prove that $R_t \in L^1(Q^-)$. Finally, equation (6.10) provides $E^{Q^+}[R_t | \mathcal{F}_{t-1}] \in \hat{C}^+$, $E^{Q^-}[R_t | \mathcal{F}_{t-1}] \in -\hat{C}^-$, P.a.s and therefore $Q^+ \in \mathcal{P}^+$ and $Q^- \in \mathcal{P}^-$. 

### 6.2 Proof of Theorem 3.2

The proof is similar to that of Theorem 3.1; then we only highlight the main changes.

The implication (ii) $\implies$ (i) follows from the supermartingale property of any wealth process $X^{x,\pi}$ under some fixed $Q \in \mathcal{P}^+ \cap \mathcal{P}^-$ (see Remark 6.1).

To prove the converse implication, we consider for any $t = 1, \ldots, T$ the sets:

$$\mathcal{B}_t = \mathcal{B}_t^+ \cup \mathcal{B}_t^- = \left\{ U \in L^0_t : \exists(x, \nu) \in \mathcal{A}_{t-1}, x \nu^* \delta_t \geq U \right\}.$$
It is clear that $\mathcal{B}_t$ is a cone of $L^0_t$. By Assumption 3.3, it is easily checked that $\mathcal{B}_t$ is a convex set in $L^0_t$. By noting that $\text{cl}(\mathcal{B}_t \cap L^1_t) = \text{cl}(\mathcal{B}^+_t \cap L^1_t) \cup \text{cl}(\mathcal{B}^-_t \cap L^1_t)$, we deduce from Lemmas 6.2 and 6.3 that the no arbitrage condition implies $\text{cl}(\mathcal{B}_t \cap L^1_t) \cap L^1_t = \{0\}$ for all $t = 0, \ldots, T$. As in Lemma 6.4, applying Yan's theorem, we obtain the existence of $\hat{Z}_t \in L^\infty_t$, $\hat{Z}_t > 0$, $P$-a.s., for all $t = 0, \ldots, T$, such that:

$$E[\hat{Z}_t U] \leq 0, \forall U \in \mathcal{B}_t \cap L^1_t.$$ 

By similar arguments as in the end of the proof of Lemma 6.4, we can prove that $E[Z_t R_t | \mathcal{F}_{t-1}] \in \hat{C}^+ \cap -\hat{C}^-$, $P$-a.s. where $Z_t = \hat{Z}_t/(1 + |R_t|)$. By induction, we construct $Z'_t$, such that for all $t = 1, \ldots, T$, $Z'_t \in L^\infty_t$ with $Z'_t > 0$ $P$-a.s. The process $Z'$ will also have the following properties:

$$E[Z'_t \cdots Z'_T] = 1 \text{ and } E[Z'_t \cdots Z'_T R_T | \mathcal{F}_{t-1}] \in \hat{C}^+ \cap -\hat{C}^-$ P-a.s.

Then, we construct a probability measure $Q \sim P$ defined by:

$$\frac{dQ}{dP} = \prod_{t=1}^T Z'_t$$

and which satisfies therefore $Q \in \mathcal{P}^+ \cap \mathcal{P}^-$.

References


